

Emergence in Action: Particles as Field Structures

Part 4: From Interactions Between Classical Fields to Interactions Between Particles

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Introduction

In this part, we will examine concepts and phenomena that are only very rarely discussed in a classical framework. These concepts are usually reserved for introductions to quantum field theory. Here, however, we deliberately treat the fields as purely classical. The aim is to give the reader a more intuitive interpretation of these phenomena before studying them later in a quantum framework.

The concepts we will address in turn are the following:

- Interactions mediated by fields.
- Range of interactions.
- Field excitation thresholds.
- Solitons and tachyons.
- Charged fields.
- Matter and antimatter.
- Creation and annihilation of matter.
- Emergence of relativistic effects.

Quantum theory is of course indispensable for understanding how these phenomena actually occur in the real world, but the same underlying mathematical structures often already account for them in a classical framework.

Before addressing these phenomena, we will make a somewhat abstract detour. We will introduce a very useful formalism: the Lagrangian formalism. This formalism makes it possible to formulate a physical theory in a way that highlights symmetries and conserved quantities. It is extremely useful in classical field theory and completely indispensable in quantum field theory. In the spirit of this series of articles, we will keep the technical machinery to a minimum and focus on conceptual understanding. We will therefore not spell out every derivation and will sometimes move fairly quickly to the important results when doing so does not compromise the deeper physical understanding.

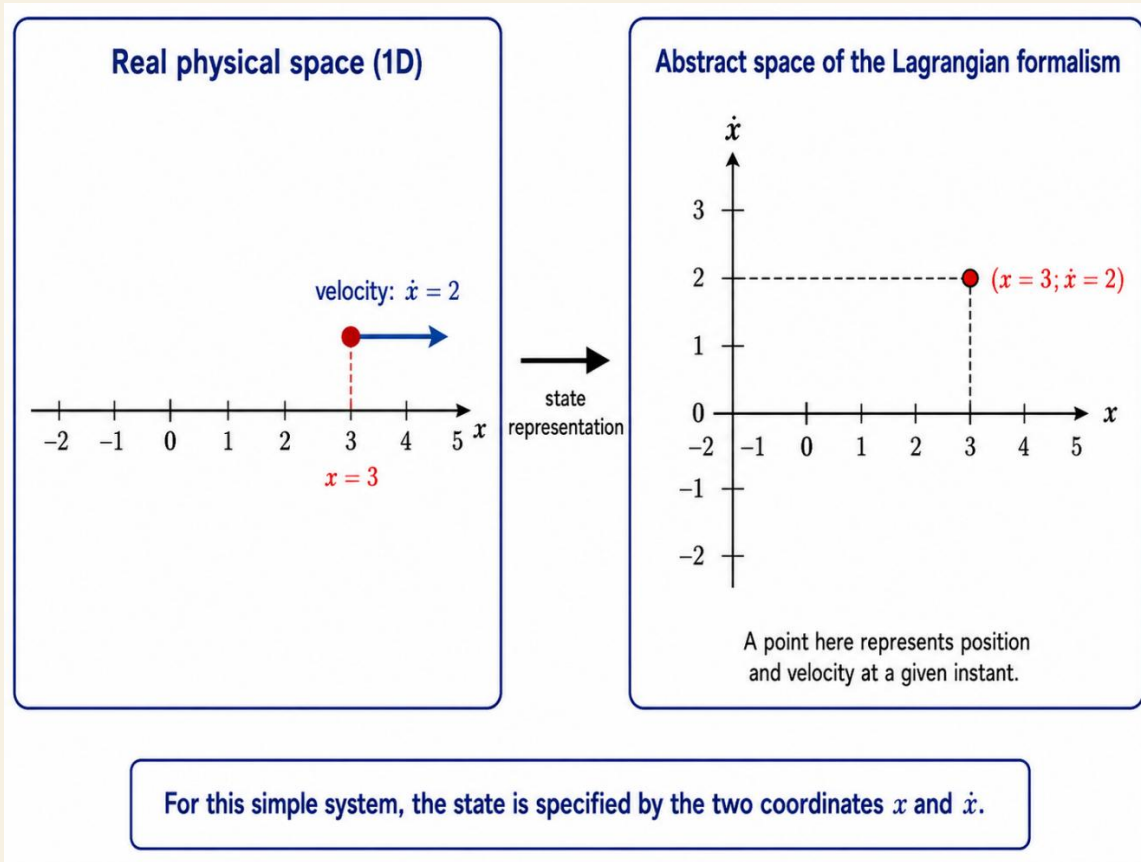
I. Lagrangian Formalism

The Lagrangian formulation of classical mechanics is an alternative to the Newtonian formulation. It allows us to derive the equations of motion using whichever variables are most convenient, while making the symmetries of the system and its conserved quantities explicit.

I.1 The framework: state space

In the Lagrangian formalism, we do not follow the system in physical space but in an abstract space: state space. A system with N degrees of freedom has a $2N$ -dimensional state space. Its state is represented by a point in this space. To make this more concrete, consider a point particle moving in

one-dimensional physical space. Its position in physical space is represented by the variable x . In the Lagrangian formalism, we can construct a two-dimensional state space: the x -axis corresponds directly to position in space, and the $\dot{x}$ -axis corresponds to the particle velocity. A point in this space (for example: $x=3$; $\dot{x}=2$) directly represents the position and velocity of our particle at a given instant. The Lagrangian formalism provides the equations of motion for this point in state space.



I.2 The principle of stationary action and the Euler-Lagrange equation

At the origin of the Lagrangian formalism lies the principle of stationary action, which states:

Consider a physical system with N degrees of freedom, whose initial state is represented by a point in the space $(x_{i1}, x_{i2}, \dots, x_{iN})$ and whose final state is represented by another point $(x_{f1}, x_{f2}, \dots, x_{fN})$. There exists a functional (a function of functions) $S(\dots x_N(t), \dots \dot{x}_N(t))$, called the action, which is stationary under any infinitesimal change in the real trajectory of the system in the space $(\dots x_N(t))$. To simplify the notation, in what follows we will consider a particle moving in one-dimensional space and write $\dot{x} = v$. The action is defined by:

$$S(x(t), v(t)) = \int_{t_i}^{t_f} L(x(t), v(t)) dt$$

where L is the Lagrangian of the system.

The stationarity condition for the action is written $\delta S = 0$ for any infinitesimal variation of the system trajectory in configuration space. For this condition to hold, one can show that the Lagrangian must satisfy the Euler-Lagrange equation:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial v} \right) - \left(\frac{\partial L}{\partial x} \right) = 0$$

This equation is equivalent to Newton's second law. To see this, we need to specify what the Lagrangian is. In classical mechanics, the Lagrangian is simply defined as the difference between kinetic energy and potential energy:

$$L = \frac{1}{2}mv^2 - V(x)$$

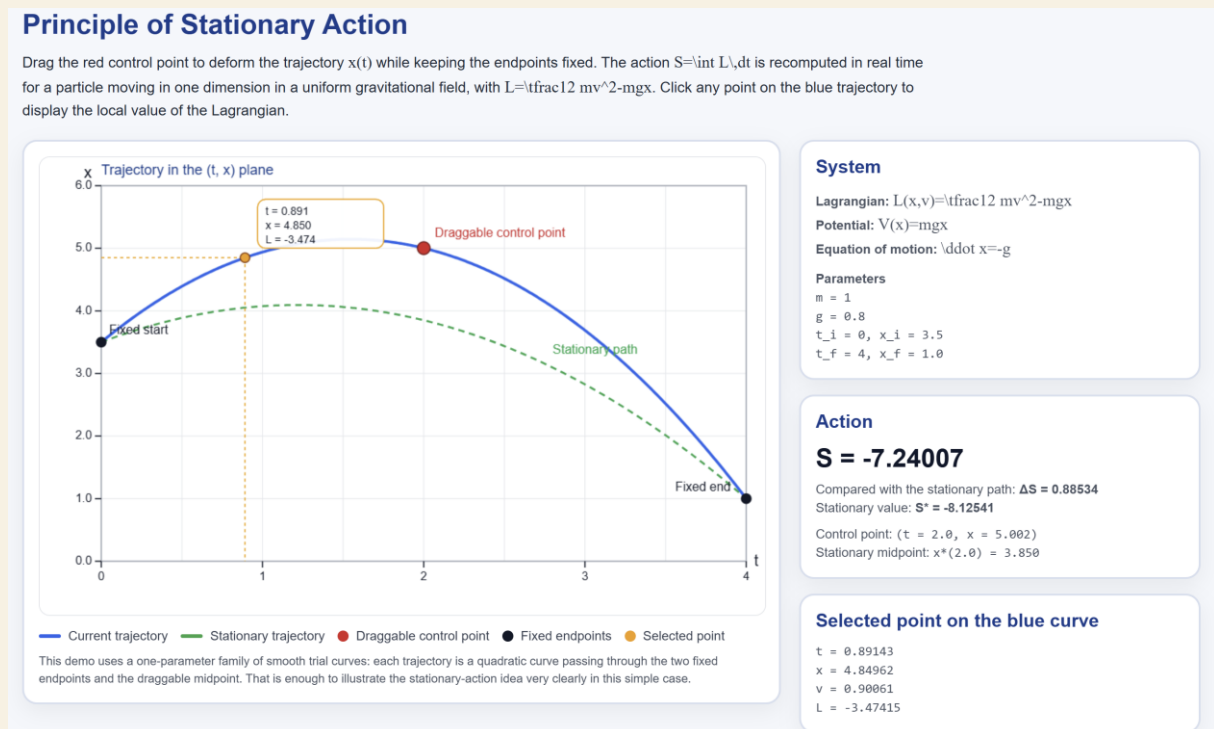
Substituting the expression for L into the Euler-Lagrange equation gives directly:

$$m \frac{dv}{dt} = -\frac{\partial V(x)}{\partial x}$$

which is indeed Newton's second law.

If the system has N degrees of freedom, there are N independent Euler-Lagrange equations, each describing the evolution of a dynamical coordinate and its time derivative in state space.

In the diagram below, the system is an object subject to a linear potential (for example, a constant gravitational force). We may imagine an object launched vertically upward at $t=0$ from 3.5 metres above the ground and located 1 metre above the ground at $t=4$. The question is then which trajectory the object will actually follow.



The blue trajectory represents an arbitrary path. The value of the Lagrangian at the yellow point is shown in the yellow box ($L=-3.474$). The action S along this trajectory is computed and displayed prominently on the right ($S=-7.24007$).

The green trajectory represents the path the object will actually follow. The value of S along this path is the minimum value: -8.12541 . The closer the blue trajectory comes to the green one, the more stable the value of the action becomes and the closer it gets to the minimum. Solutions of the Euler-Lagrange equation correspond to trajectories that make the action stationary (in this case, the action is therefore minimal).

You can experiment with [the animation on the article presentation page](#).

Note: the trajectory can be modified from only one point. The animation therefore gives access to only one family of trajectories. The principle of stationary action should be understood as a stationarity principle over all conceivable families of trajectories.

I.3 Conjugate momentum and Hamiltonian

There is an alternative formalism to the Lagrangian one: the Hamiltonian formalism. We will not develop it here, but will introduce only two of its concepts: conjugate momentum and the Hamiltonian.

The conjugate momentum of a dynamical variable $x(t)$ is defined by:

$$p(t) = \frac{\partial L}{\partial v}$$

It corresponds to momentum in classical mechanics. The Hamiltonian, meanwhile, is defined by

$$H(x, p) = pv - L$$

which, for our one-dimensional system, gives:

$$H(x, p) = \frac{p^2}{2m} + V(x)$$

The Hamiltonian corresponds to the energy of the system expressed in terms of conjugate momentum rather than velocity.

I.4 First links between symmetries and conserved quantities

There is an important theorem that we will soon encounter in the Lagrangian formulation of field theory: Noether's theorem. It states that every symmetry of the system (that is, every invariance of the Lagrangian under a transformation) corresponds to a conserved quantity. Without treating the theorem in full generality here, we can very quickly derive two of its results:

- Spatial translation symmetry and conservation of momentum

If $L(x, v)$ does not explicitly depend on x , then according to the Euler-Lagrange equation:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial v} \right) = 0$$

And therefore:

$$\frac{dp}{dt} = 0$$

which simply means that $p = \text{constant}$.

- Time-translation symmetry and conservation of energy

If $L(x, v)$ does not explicitly depend on t , then from the definition of the Hamiltonian:

$$\frac{d}{dt} H(x, p) = 0$$

And therefore:

$$H = \text{cste}$$

The Lagrangian formalism therefore provides a way to ensure that a theory respects the conservation of certain quantities.

1.5 Hamilton-Jacobi equation

The Hamilton-Jacobi equation can be derived starting from the formulation of the action

$$S(x(t), v(t)) = \int_{t_i}^{t_f} L(x(t), v(t)) dt$$

If we take $dS = L dt$, then

$$\frac{dS}{dt} = L$$

Moreover, by definition:

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial x} \frac{dx}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial x} v$$

And we recall that $H(x, p) = pv - L$

And therefore: $L = pv - H(x, p)$

We thus have, on the one hand, $\frac{dS}{dt} = pv - H(x, p)$ and, on the other, $\frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial x} v$

By identification, we have: $p = \frac{\partial S}{\partial x}$ and $\frac{\partial S}{\partial t} = -H(x, \frac{\partial S}{\partial x})$

The Hamilton-Jacobi equation is:

$$\frac{\partial S}{\partial t} = -H(x, \frac{\partial S}{\partial x})$$

For a particle in one-dimensional space, we have: $H(x, \frac{\partial S}{\partial x}) = \frac{1}{2m} (\frac{\partial S}{\partial x})^2 + V(x)$

and therefore: $\frac{\partial S}{\partial t} + \frac{1}{2m} (\frac{\partial S}{\partial x})^2 + V(x) = 0$

This equation may look like nothing more than a complicated formulation of mechanical energy as the sum of kinetic and potential energy. It is nevertheless interesting because it has the form of an equation for the phase S of a wave .. The main interest of this formulation is that it turns the search for a trajectory into the solution of a phase field $S(x, t)$. The particle trajectory then becomes analogous to a light ray in geometrical optics: at every point it is perpendicular to the wavefronts of constant action.

If the potential is time-independent, $\frac{\partial S}{\partial t} = -E$, where E is a constant representing the energy of the system. The Hamilton-Jacobi equation then gives directly:

$$\frac{\partial S}{\partial x} = \pm \sqrt{2m(E - V(x))}$$

Let $W(x)$ be the part of S that depends on x .

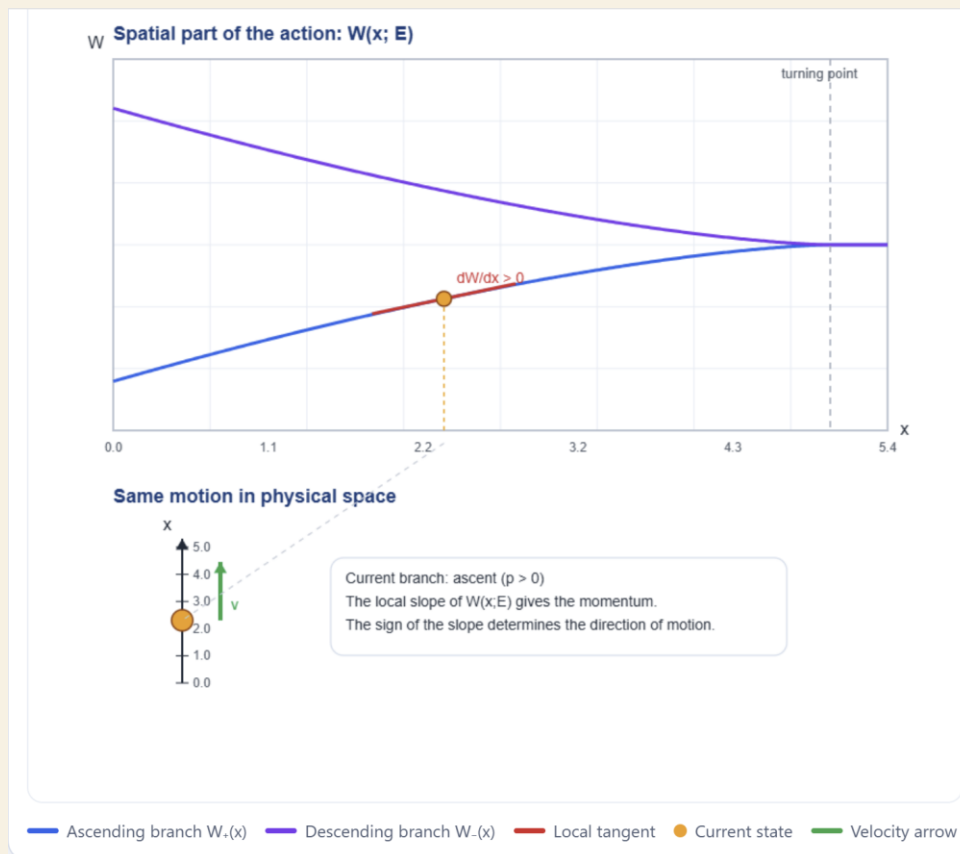
We therefore have $\frac{\partial S}{\partial x} = \frac{\partial W}{\partial x} = \pm \sqrt{2m(E - V(x))}$

It is enough to integrate over x to obtain the spatial form of S . Since $p = mv = \frac{\partial S}{\partial x} = \frac{\partial W}{\partial x}$, we infer that the velocity of the object is tangent to $W(x)$. In other words, the spatial component of S provides a velocity field for the object. The phase wave S appears to guide the particle.

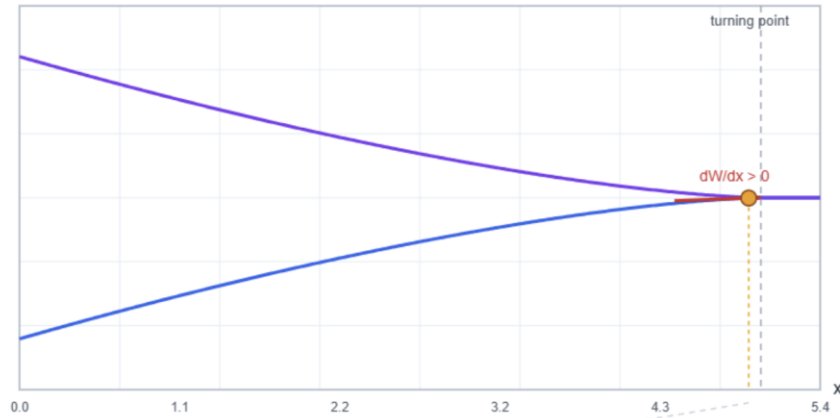
Consider the case of a potential linear in x , for example $V(x) = gx$.

Integrating gives:

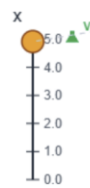
$$W(x) = \mp \frac{2\sqrt{2m}}{3g} (E - gx)^{3/2} + C$$



W Spatial part of the action: $W(x; E)$



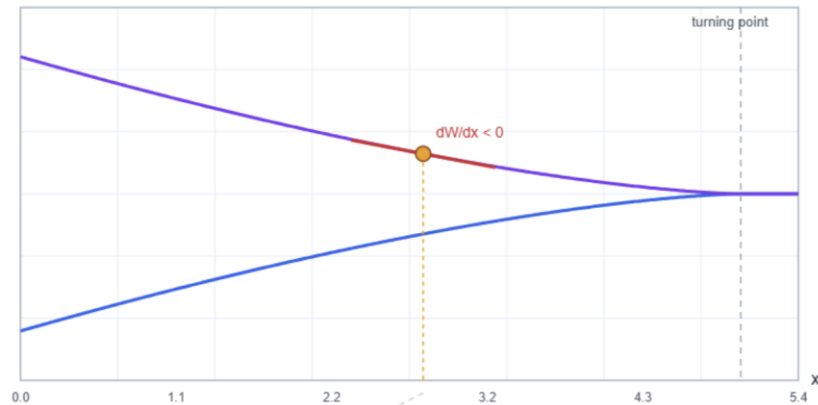
Same motion in physical space



Current branch: ascent ($p > 0$)
The local slope of $W(x; E)$ gives the momentum.
The sign of the slope determines the direction of motion.

— Ascending branch $W_+(x)$ — Descending branch $W_-(x)$ — Local tangent — Current state — Velocity arrow

W Spatial part of the action: $W(x; E)$



Same motion in physical space



Current branch: descent ($p < 0$)
The local slope of $W(x; E)$ gives the momentum.
The sign of the slope determines the direction of motion.

— Ascending branch $W_+(x)$ — Descending branch $W_-(x)$ — Local tangent — Current state — Velocity arrow

In the images from the animation above, the slope of $W(x)$ at the yellow point is proportional to the velocity of the particle (whose trajectory in physical space is shown below). During the upward phase,

the object simply follows the spatial derivative of the positive solution for $W(x)$ until the velocity reaches zero. The object then begins its downward trajectory and follows the spatial derivative of the negative solution for $W(x)$. The animation is available [on the article presentation page](#).

We can show one last thing using this equation. This time, consider a free particle ($V=0$). The equation becomes:

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \left(\frac{\partial S}{\partial x} \right)^2 = 0$$

If we assume that the particle has a fixed energy E , we can write:

$$S(x, t) = W(x) - Et$$

The equation then becomes:

$$E = \frac{1}{2m} \left(\frac{\partial W}{\partial x} \right)^2$$

or

$$\frac{\partial W}{\partial x} = \sqrt{2mE} + cste$$

Let us set $p = \sqrt{2mE}$

$S(x, t)$ becomes: $S(x, t) = px - Et + cste$

to be compared with the expression for the phase of a wave:

$$\phi(x, t) = kx - \omega t + cste$$

The correspondence between action and phase, conjugate momentum and wave number, and energy and angular frequency then becomes transparent.

I.6 Euler equation

Starting from the Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \left(\frac{\partial S}{\partial x} \right)^2 + V = 0$$

If we define the particle velocity as $v = \frac{\partial S}{\partial x}$, then by applying the operator $\frac{\partial}{\partial x}$ to the Hamilton-Jacobi equation we obtain the Euler equation:

$$\frac{\partial v}{\partial t} + \left(v \cdot \frac{\partial}{\partial x} \right) v = -\frac{1}{m} \frac{\partial V}{\partial x}$$

which is the fluid-mechanics equation for a fictitious fluid defining the velocity field of a particle subject to a potential V . It is in fact equivalent to Newton's second law: $\frac{\partial v}{\partial t} + \left(v \cdot \frac{\partial}{\partial x} \right) v$ is simply equal to $\frac{dv}{dt}$. Here, however, velocity is a field defined throughout space. The term $-\frac{1}{m} \frac{\partial V}{\partial x}$ is simply equal to $\frac{F}{m}$.

We will see that equations with the same form as the Euler equation can be derived from field equations, which will allow us to connect field dynamics directly with the dynamics of the

corresponding particles. If, for example, we obtain the Euler equation for a field, its velocity field will represent the velocities of the different possible wave packets.

1.7 Lagrangian formalism for fields

For a field, we define a Lagrangian density $\mathcal{L}$ such that the total Lagrangian is the spatial integral of this density:

$$L = \int d x \mathcal{L} \left(\phi(x, t), \frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x} \right)$$

The action S then becomes an integral over spacetime:

$$S = \int d t \int d x \mathcal{L} \left(\phi(x, t), \frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x} \right)$$

We seek the field $\phi(x)$ that makes the action stationary, ($\delta S = 0$), under a small variation $\delta\phi(x)$ that vanishes at the boundaries.

One can then show that the density $\mathcal{L}(\phi, \partial_\mu \phi)$ must satisfy the fundamental equation:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \phi}{\partial t} \right)} \right) - \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \phi}{\partial x} \right)} \right) = 0$$

This is the Euler-Lagrange equation for fields. It allows us to derive the wave equations governing the fields. For a system made of several fields, there are as many Euler-Lagrange equations as there are independent fields.

1.8 Choosing Lagrangians

With the Lagrangian formalism, the decisive step in constructing a theory is to find the right Lagrangian. Once the Lagrangian has been chosen, the properties of the system are fixed. In field theory, the Lagrangian formalism has another advantage: the Lagrangian density is defined as a relativistic scalar, that is, a Lorentz invariant. All terms that make up the Lagrangian must therefore themselves be Lorentz scalars, which strongly constrains the possibilities. In addition, requiring the symmetries that guarantee the corresponding conservation laws provides another powerful constraint.

But although symmetries guide the theorist, there is no magic recipe for ensuring that a chosen Lagrangian is the correct one. Once a choice has been made, one must derive the observable consequences of the theory and compare them with experimental results.

1.9 Noether's theorem: from symmetry to conservation

Noether's theorem is one of the pillars of modern physics. It generalizes the observations above by establishing a rigorous mathematical link: every continuous symmetry of the Lagrangian corresponds to a conserved quantity.

A system is said to possess a symmetry if the Lagrangian density $\mathcal{L}$ remains invariant (up to a total divergence) under a continuous transformation of the fields depending on an infinitesimal parameter ϵ :

$$\phi(x) \rightarrow \phi(x) + \epsilon \Delta \phi(x)$$

Noether's theorem states that, if such a symmetry exists, one can construct a relativistic four-vector called the Noether current J^μ , defined by:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \phi}{\partial \mu} \right)} \Delta \phi$$

where μ denotes the spacetime components of the vector (t, x in our one-dimensional models).

(Note: if the transformation also affects the spacetime coordinates, an additional term involving the energy-momentum tensor must be added.)

This current satisfies the continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0$$

where ρ is the time component of the Noether current and j its spatial component.

Locally, this equation means that "what leaves a small volume equals the change in what remains inside it".

Integrating the time component of the current (ρ , also called the charge density) over all space gives a global charge Q :

$$Q = \int \rho dx$$

The continuity equation guarantees that this charge remains constant in time: $\frac{dQ}{dt} = 0$

Two broad types of symmetry are generally distinguished:

Type of symmetry	Transformation	Conserved quantity
Spacetime	Time translations	Energy
	Spatial translations	Momentum
	Spatial rotations	Angular momentum
Internal	Phase transformations (gauge)	Charges (electric, colour, etc.)

II. Interaction between quasi-particles

In Part 3, we showed that under certain conditions a wave packet can temporarily behave like a particle obeying relativistic or Newtonian dynamics. The dynamics of such a wave packet, however, were studied only in the presence of a fixed external field.

In ordinary physical theories, however, fields are themselves physical systems with their own dynamics.

In modern formulations of classical physics, fields are emitted by particles, while particles are themselves affected by fields. Particles do not interact directly with one another; they interact through fields that act as mediators.

In quantum field theory, the situation is described differently. Fundamentally, despite the picture often used in pedagogical presentations, quantum field theory describes neither interactions between particles nor even interactions between particles and fields, but interactions between fields.

Yet we do not need to abandon the classical framework in order to understand how the familiar description of particle interactions can emerge from a purely field-based ontology. The models we will study are intended precisely to illuminate this emergence.

II.1 A mechanical model of interaction between three fields

To introduce the idea of a mediated interaction between quasi-particles in a simple way, we will consider a mechanical model containing three coupled fields, denoted $A(x, t)$, $B(x, t)$ and $C(x, t)$.

Each of these fields can be represented mechanically by a chain of identical small masses connected by springs. The fields A and C will carry wave packets, while the field B will act as the mediator field.

The physical idea of the model is the following. A wave packet present in A locally perturbs B . This perturbation then propagates through B and influences the dynamics of a wave packet located in C . The apparent interaction between the two wave packets is therefore not a direct action of one on the other, but the emergent manifestation of an interaction dynamics between fields.

In the continuum approximation, the dynamics of the three fields are described by the following equations:

$$\begin{aligned}\frac{\partial^2 A}{\partial t^2} &= c_A^2 \frac{\partial^2 A}{\partial x^2} - \Omega_A^2 A + g_{AB}(B - A) \\ \frac{\partial^2 B}{\partial t^2} &= c_B^2 \frac{\partial^2 B}{\partial x^2} - \Omega_B^2 B + g_{AB}(A - B) + g_{BC}(C - B) \\ \frac{\partial^2 C}{\partial t^2} &= c_C^2 \frac{\partial^2 C}{\partial x^2} - \Omega_C^2 C + g_{BC}(B - C)\end{aligned}$$

Mechanical interpretation of the different terms

Each term in these equations has a simple mechanical interpretation.

The term

$$c^2 \frac{\partial^2}{\partial x^2}$$

comes from the springs connecting each mass to its neighbours along the chain. This is the term that allows disturbances to propagate through the field as waves.

The term

$$-\Omega^2 A$$

represents a local restoring force toward equilibrium. Mechanically, it corresponds to a spring connecting each mass to a fixed support. In the analogy with physical fields, this term plays the role of a mass term.

The coupling terms between fields come from springs connecting corresponding masses in the different chains. For example, the term

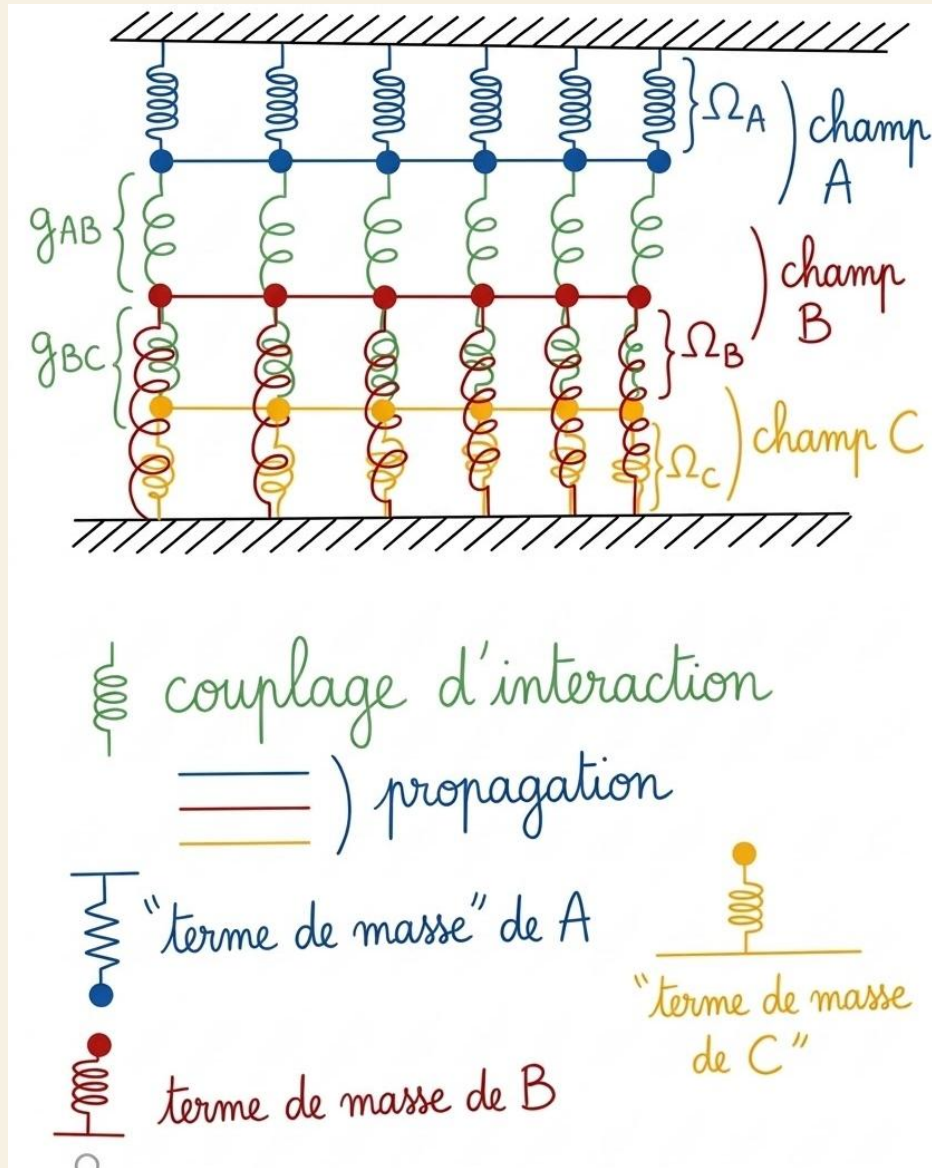
$$g_{AB}(B - A)$$

means that the mass belonging to chain A is connected to the corresponding mass in chain B . The coupling spring then exerts a force proportional to the difference in displacement between the two masses.

Thus:

- if $B > A$, the spring pulls A upward;
- if $A > B$, it pulls A downward.

The same reasoning applies to the coupling between B and C .



Rewriting the equations

It is instructive to expand the coupling terms.

We then obtain:

$$\begin{aligned}
 \frac{\partial^2 A}{\partial t^2} &= c_A^2 \frac{\partial^2 A}{\partial x^2} - \Omega_A^2 A + g_{AB} B - g_{AB} A \\
 \frac{\partial^2 B}{\partial t^2} &= c_B^2 \frac{\partial^2 B}{\partial x^2} - \Omega_B^2 B + g_{AB} A - g_{AB} B + g_{BC} C - g_{BC} B \\
 \frac{\partial^2 C}{\partial t^2} &= c_C^2 \frac{\partial^2 C}{\partial x^2} - \Omega_C^2 C + g_{BC} B - g_{BC} C
 \end{aligned}$$

Grouping together the terms proportional to the fields themselves gives:

$$\begin{aligned}
 \frac{\partial^2 A}{\partial t^2} &= c_A^2 \frac{\partial^2 A}{\partial x^2} - (\Omega_A^2 + g_{AB}) A + g_{AB} B \\
 \frac{\partial^2 B}{\partial t^2} &= c_B^2 \frac{\partial^2 B}{\partial x^2} - (\Omega_B^2 + g_{AB} + g_{BC}) B + g_{AB} A + g_{BC} C
 \end{aligned}$$

$$\frac{\partial^2 C}{\partial t^2} = c_C^2 \frac{\partial^2 C}{\partial x^2} - (\Omega_C^2 + g_{BC})C + g_{BC}B$$

Emergence of effective masses

This rewriting reveals an important point: the coupling springs do not merely generate an interaction between fields. They also modify the local restoring terms.

We can therefore define renormalized effective masses:

$$\begin{aligned}\Omega_{A,eff}^2 &= \Omega_A^2 + g_{AB} \\ \Omega_{B,eff}^2 &= \Omega_B^2 + g_{AB} + g_{BC} \\ \Omega_{C,eff}^2 &= \Omega_C^2 + g_{BC}\end{aligned}$$

The equations then become:

$$\begin{aligned}\frac{\partial^2 A}{\partial t^2} &= c_A^2 \frac{\partial^2 A}{\partial x^2} - \Omega_{A,eff}^2 A + g_{AB}B \\ \frac{\partial^2 B}{\partial t^2} &= c_B^2 \frac{\partial^2 B}{\partial x^2} - \Omega_{B,eff}^2 B + g_{AB}A + g_{BC}C \\ \frac{\partial^2 C}{\partial t^2} &= c_C^2 \frac{\partial^2 C}{\partial x^2} - \Omega_{C,eff}^2 C + g_{BC}B\end{aligned}$$

Interpretation: a first form of renormalization

An important phenomenon therefore appears: introducing interactions modifies the effective parameters of the system.

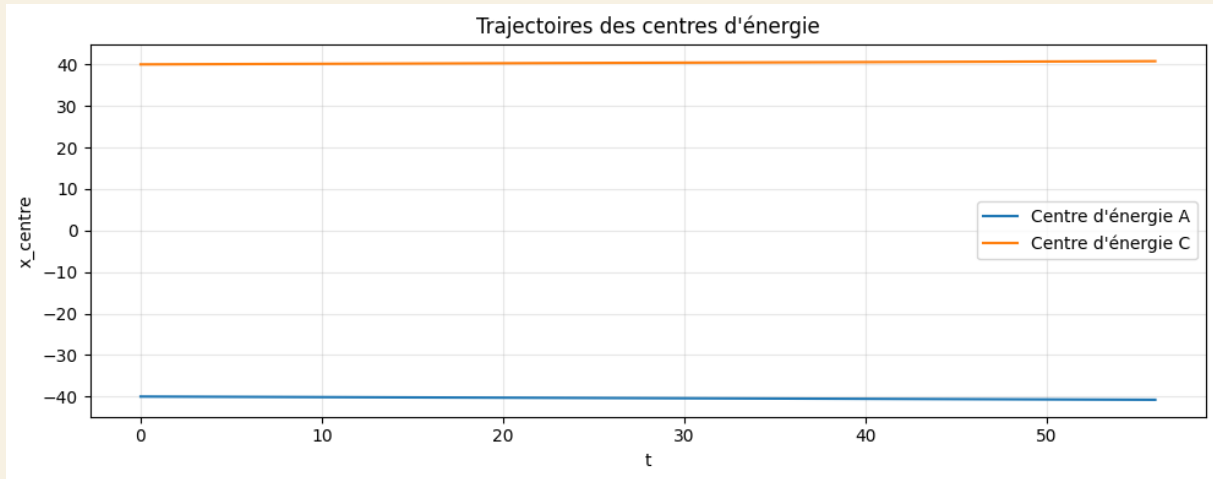
The springs connecting the different chains do more than create an influence between fields. They also modify the effective stiffness experienced by each field. In other words, the physical masses appearing in the equations of motion no longer correspond to the "bare" physical masses that each field would have if it were not coupled to the others.

This phenomenon is a very simple version of what, in more elaborate field theories, is called parameter renormalization: interactions modify the effective values of the constants that appear in the dynamical equations.

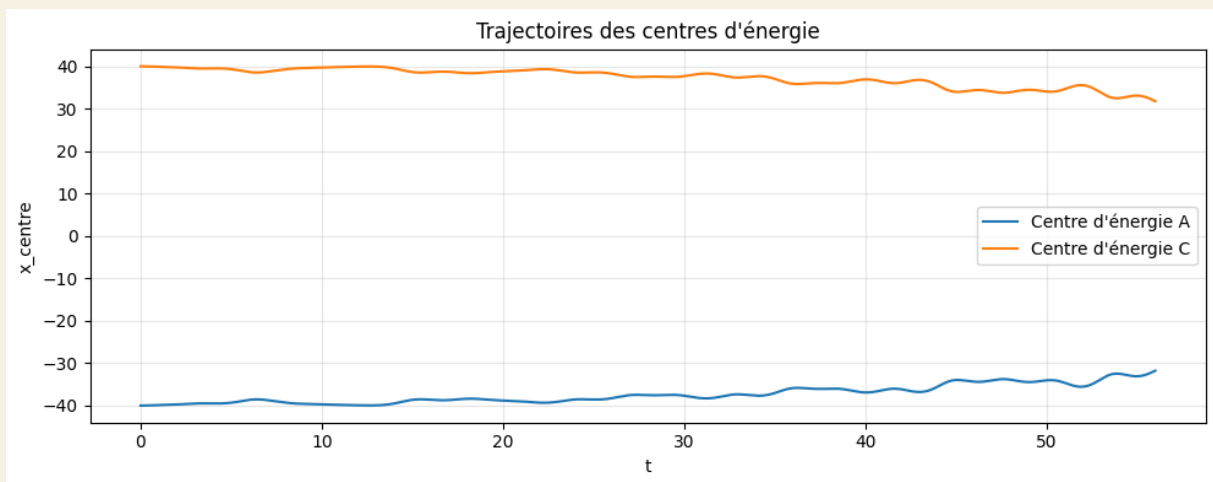
In the mechanical model considered here, this mechanism already appears transparently, without any particular mathematical difficulty.

Simulation and limitations of the model

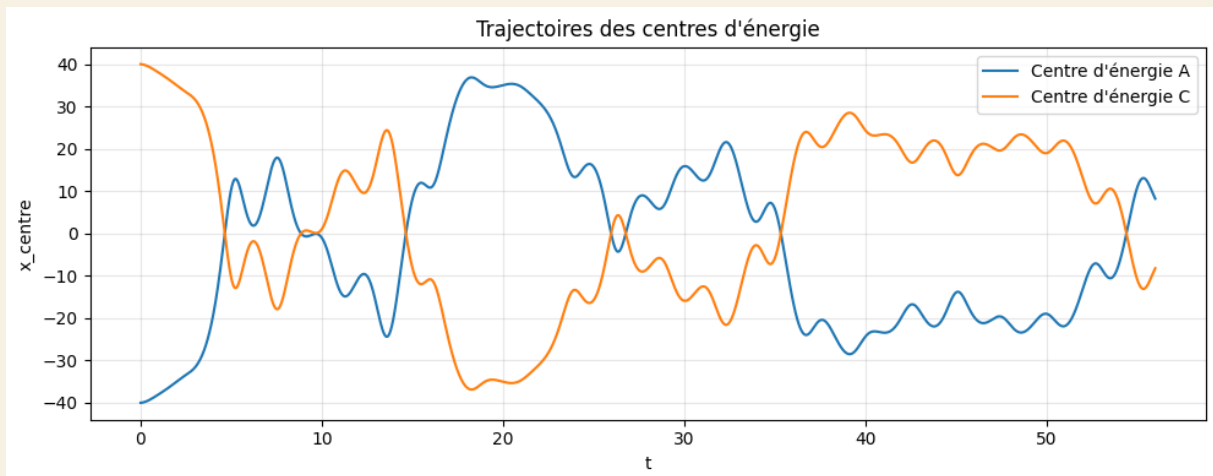
Let us test this model numerically. We choose initial conditions for fields A and C containing one wave packet in each field, separated by some distance, both with zero group velocity. Field B is assigned a high propagation coefficient. For the moment, we give it a zero mass parameter. We first track the trajectories of the energy centres of fields A and C when their coupling coefficients to B are zero.



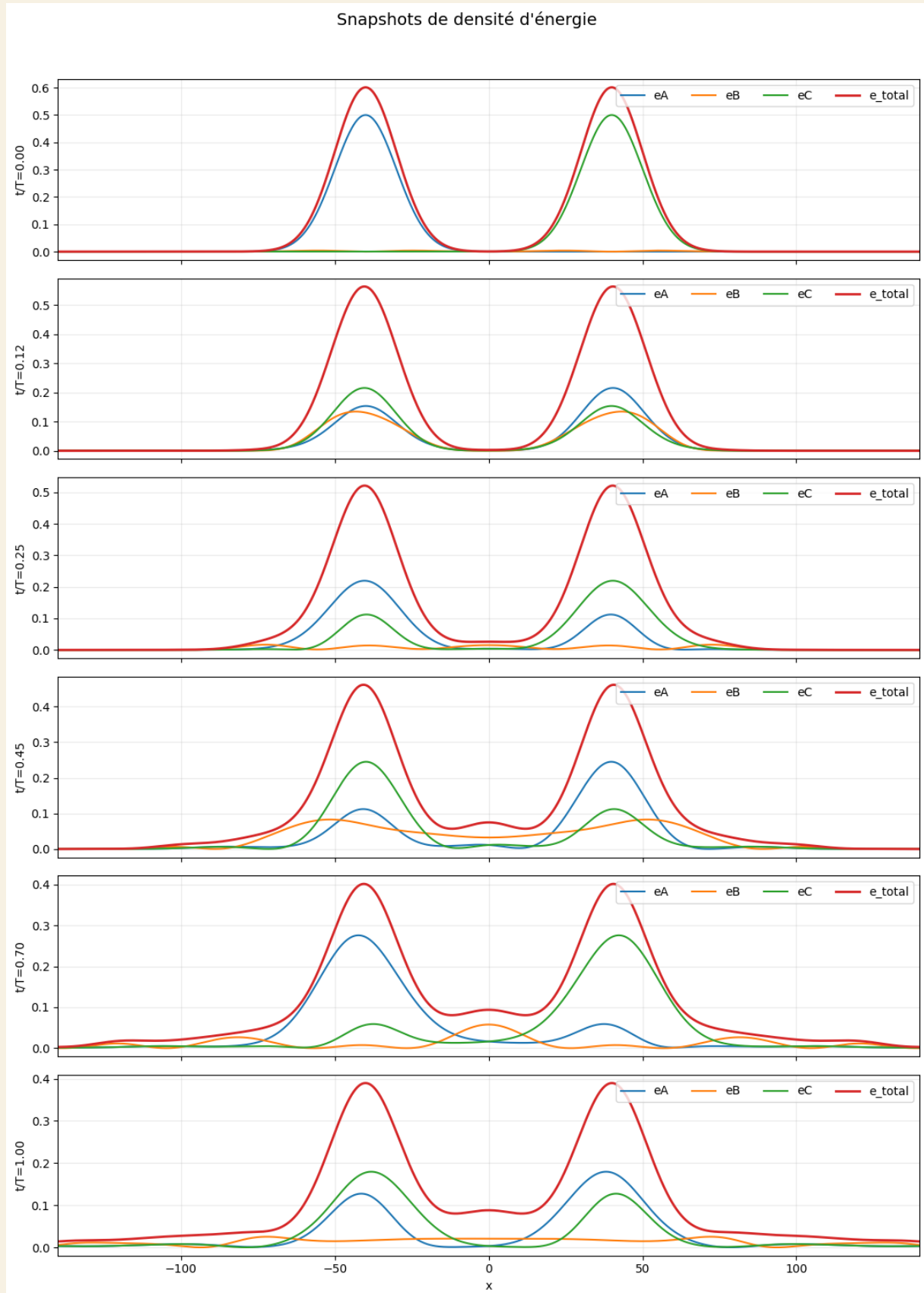
The energy centres remain perfectly stationary: as expected, there is no interaction between fields A and C. Now let us switch on the coupling and set it to a small value:



We see that the energy centres slowly move closer together. B appears to mediate an attractive interaction between the two centres of energy. Let us now increase the coupling:



Increasing the coupling therefore produces a stronger apparent interaction between the two energy centres. Yet this picture is extremely misleading. The two energy centres do not in fact interact! Looking more closely at the dynamics of the system, we see that the centres of the wave packets remain stationary. The wave packets disappear and re-form at two locations, and it is this process that is responsible for the motion of the energy centres.



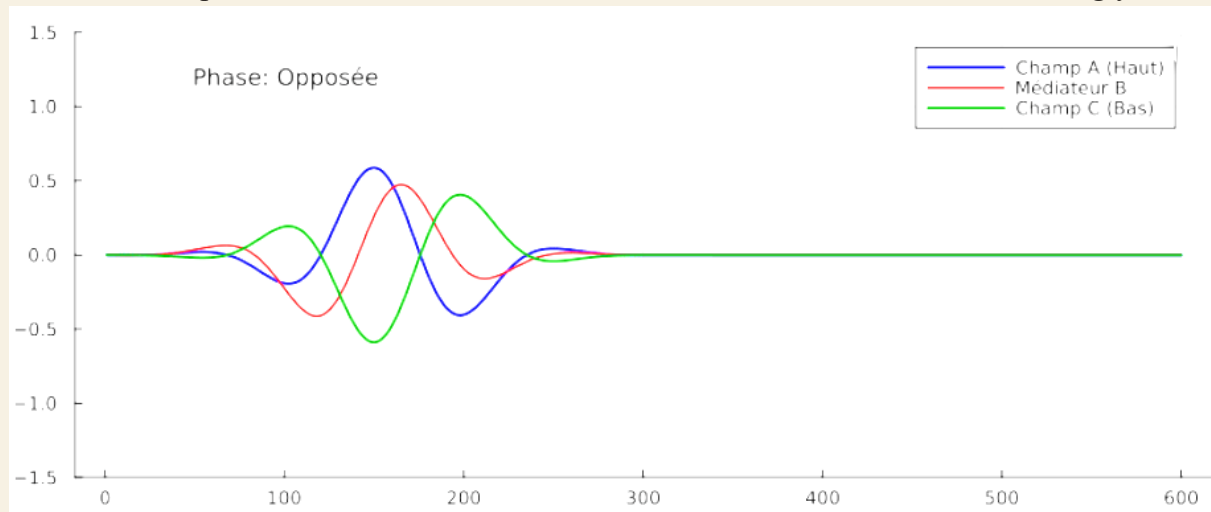
We are not dealing with the dynamics of two quasi-particles exerting a mutual force. If we wish to keep using particle language, this phenomenon does not mimic the mediation of an effective force between two particles, but rather a process of particle creation and annihilation.

The excitation of A on the left-hand site gradually transfers its energy, through B, to C and thereby creates an excitation of C at the same site. A symmetric reverse process occurs on the right-hand site.

Because the energy transfer is not instantaneous, the energy centres of the two fields appear to follow somewhat erratic trajectories around the total energy centre of the system. But the energy flows are from one field to another, not across the horizontal x-direction.

The real interest of the model

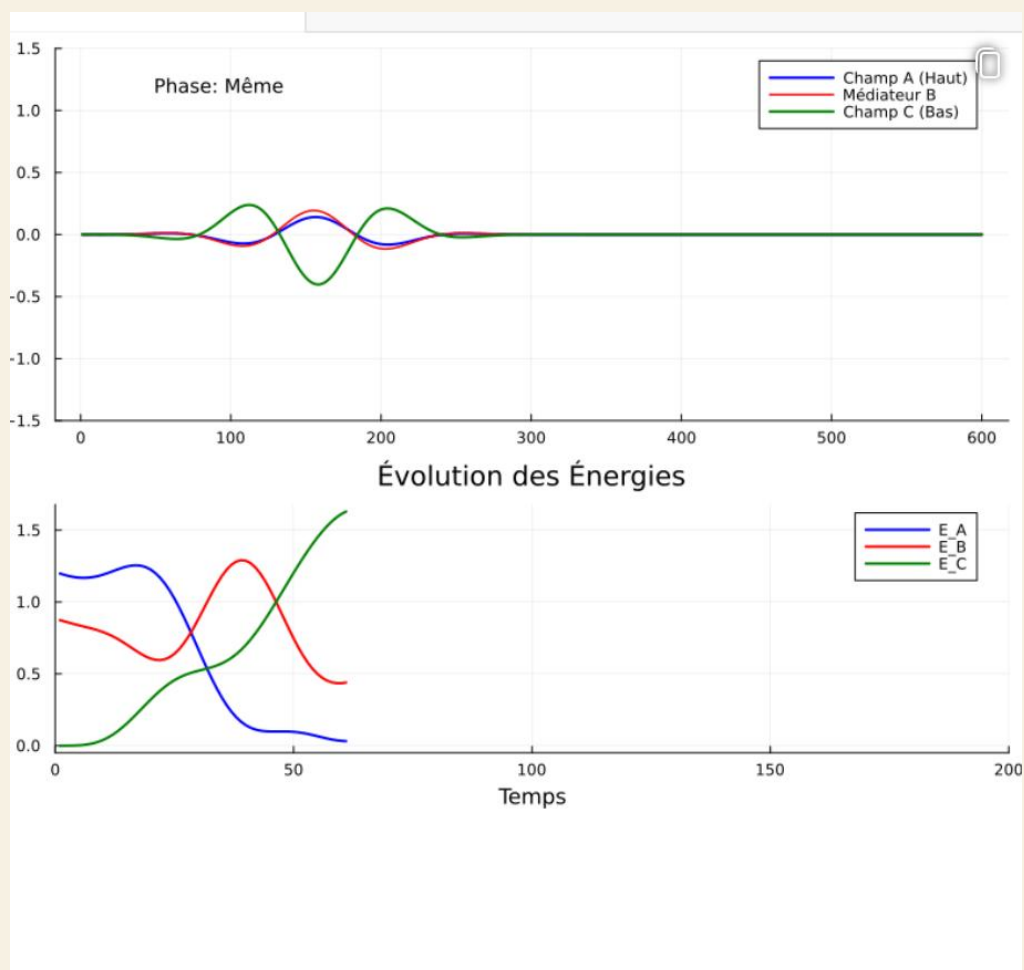
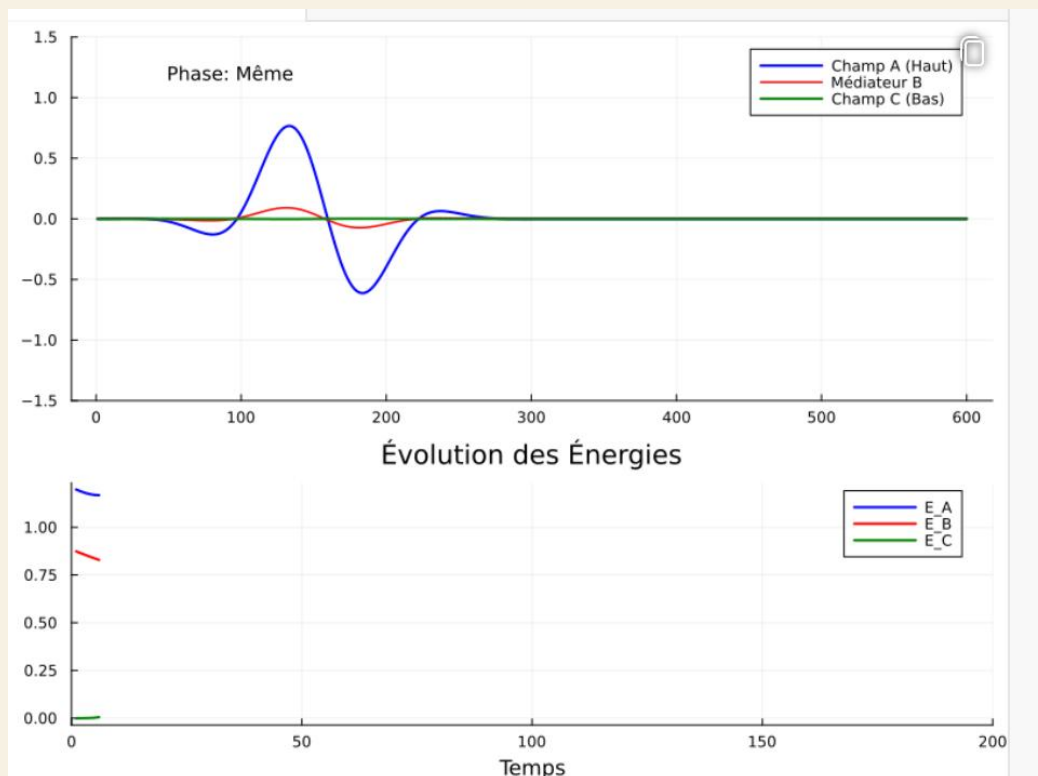
The main purpose of this model was to illustrate intuitively what particle interactions mean in field theory: exchanges of energy between coupled fields. This particular mechanical model, however, does not reproduce particle interactions of the kind familiar from Newtonian mechanics. It can nevertheless offer a glimpse of a real phenomenon in particle physics: the oscillation of certain particles between two states. Some neutral mesons, as well as neutrinos, oscillate between several states. To take a concrete example, the meson K_0 occurs in nature in two forms: the kaon K_{0S} and the kaon K_{0L} . The two kaon states are themselves superpositions of a particle and its antiparticle. They are not elementary particles, but within effective field theories each of these particles can be modelled by a different field. Our model can provide a simplified version of kaon physics. We can, for example, associate field A with a particle and field C with its antiparticle. The kaon state K_{0S} then corresponds to a state in which one wave packet is present in field A and another in field C, with the two oscillating out of phase. In that case, the mediator is strongly excited.

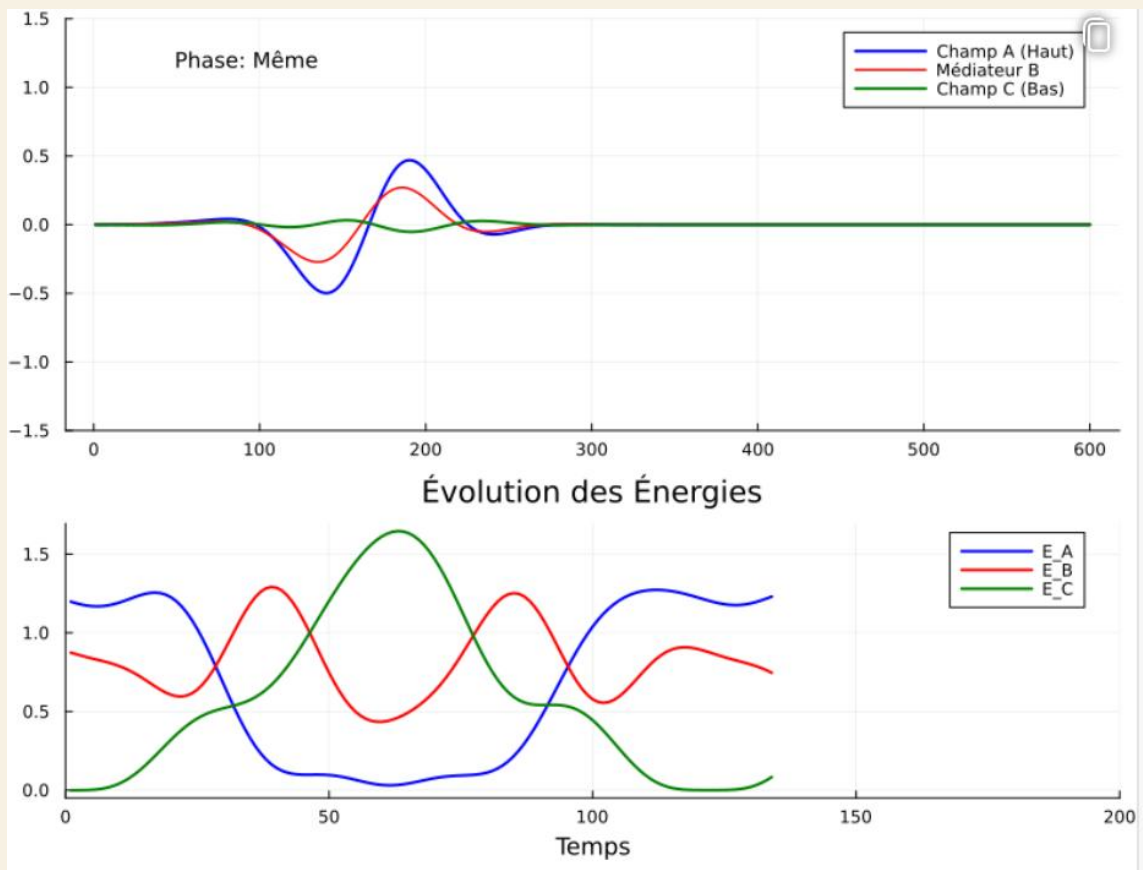


The

kaon state K_{0L} corresponds to a situation in which one packet is present in A and another in C, but this time the two oscillate in phase. In that case, the mediator field is not excited at all, even though it still mediates the coupling between the two fields.

If, on the other hand, we begin by preparing a kaon in a "pure" state (for example, with a wave packet initially present only in field A), the two fields continually exchange energy, transforming the kaon into its antiparticle and back again.





The particle thus spends its time changing "identity", which can look very strange from a particle point of view. From a field-theory point of view, however, the process becomes much more natural: two coupled fields simply exchange energy through a third mediator field. The available energy merely moves back and forth between the two fields.

Actual kaon physics is more complex, but our model gives a useful qualitative glimpse of it. It also illustrates something fundamental about particle physics: the word force is not always an appropriate way to describe interactions. This is especially true of the weak nuclear interaction. It cannot be represented classically as two particles exerting a mutual force through a field. Fundamentally, the interaction is between fields. The term force becomes appropriate only when field dynamics produce stable structures that appear to accelerate one another.

Before moving on to field couplings that can describe more general particle interactions, we will examine an even simpler mechanical model in the next section. The goal will be to understand the fundamental role played by the mass parameter in interactions, beyond the purely inertial role already discussed in the previous chapter.

II.2 Two new roles for mass

This time we will use a mechanical model with which we are already familiar: simply a series of coupled springs that obey the Klein-Gordon equation in the continuum limit.

We will add only two elements: one spring will be coupled to a "motor" that supplies energy to the field, and another spring will be connected to a "sensor" that measures the energy received by that spring.

The field therefore acts as an intermediary between the "motor" and the "sensor". It is a mediator field. We will explore two aspects of mass that play fundamental roles in field theory, including quantum field theory: mass as the main factor determining the range of an interaction, and mass as an "energy entry threshold" that must be paid in order to excite a field. We will consider two different situations: a static one, illustrating the first aspect, and a dynamical one, illustrating the second.

Static property of mass: mass determines the interaction range of a mediator field

In this first situation, the motor simply pulls on its spring (say, the spring located at $x=0$) and holds it at a fixed displacement. That spring in turn pulls its neighbours in the same direction, but the transmitted force gradually decreases. Balancing the forces on a given spring (other than the spring pulled by the motor), we obtain the following equilibrium equation:

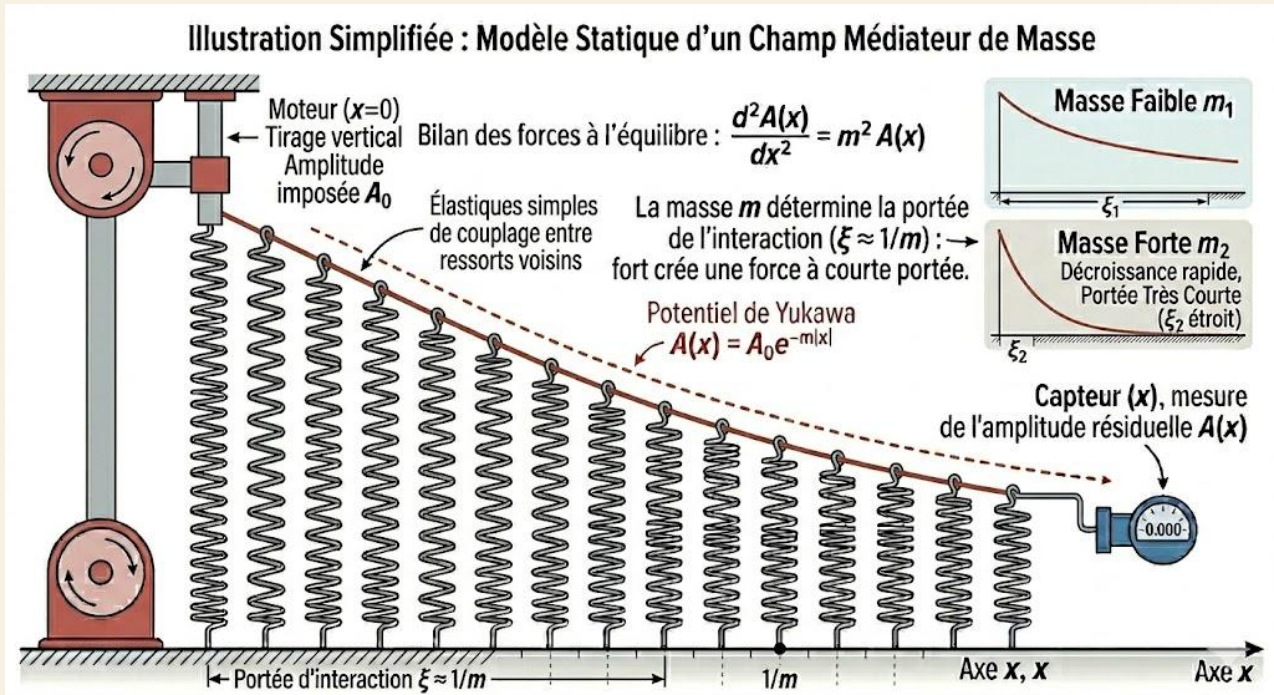
$$c^2 \frac{\partial^2 A}{\partial x^2} - \Omega^2 A = 0$$

The general solution of such an equation has the form:

$$A(x) = B e^{-\frac{\Omega}{c}x} + C e^{\frac{\Omega}{c}x}$$

Since we have stipulated that the spring is pulled at $x=0$, its maximum amplitude is reached at that point. The amplitude then depends only on the distance from $x=0$, namely $|x|$, and decreases strictly with that distance; we therefore obtain:

$$A(x) = B e^{-\frac{\Omega}{c}|x|}$$



The detector located at x therefore measures a signal that decreases exponentially, and this decay is controlled by the mass parameter Ω of the field. The function $A(x)$ that we have just derived has the standard form of a potential generated by a mediator field of mass Ω . This is called a Yukawa potential.

The field-based approach makes the interpretation of this potential more intuitive and less mysterious than the particle-based approach. In particle language, the exponential factor is interpreted as representing the propagation of a virtual particle of mass Ω , and it is difficult to claim a real understanding of this explanation without entering the machinery of Feynman diagrams in quantum field theory.

In field language, the explanation is immediate: the parameter Ω , we recall, represents a restoring coefficient that pulls the field back toward zero. The Laplacian, by contrast, represents a "force" that tends to smooth the field, that is, to oppose curvature. The equilibrium equation we wrote expresses a balance between these two effects. Near $x=0$ the field amplitude is large, so the restoring force is strong; to balance it, the field curvature must also be large. This curvature decreases in proportion to the displacement of the springs. At each point it must balance a restoring force that itself decreases linearly with the spring displacement. The curvature is therefore large near $x=0$, but decreases as we move away from $x=0$ and the amplitude becomes smaller. Hence the exponentially decaying form.

When the mass is zero, the curvature is zero: the interaction can extend indefinitely. The two classical fields detectable at macroscopic scales, the gravitational field and the electromagnetic field, have a zero mass term; this is why they remain detectable macroscopically.

Mass as an energy entry cost

We will now make the motor more active. It drives its spring periodically. Away from the spring directly attached to the motor, our equation is:

$$\frac{\partial^2 A}{\partial t^2} = c^2 \frac{\partial^2 A}{\partial x^2} - \Omega^2 A$$

We wait for the system to settle, which may take some time. Once it has done so, we have a steady regime with a constant frequency. The solution can then be written in the form:

$$A(x, t) = \alpha(x) e^{i\omega t}$$

Substituting this solution into the equation and rearranging gives:

$$\frac{\partial^2 \alpha(x)}{\partial x^2} = \frac{\Omega^2 - \omega^2}{c^2} \alpha(x)$$

It is worth pausing over this result. The form of the solutions depends on the sign of the coefficient $\sigma^2 = \frac{\Omega^2 - \omega^2}{c^2}$. If $\omega < \Omega$, then the solution $\alpha(x)$ has the form

$$\alpha(x) = \alpha_0 e^{-\sigma x}$$

In other words, the wave amplitude decreases exponentially in space. This is called an evanescent wave. Physically, the motor does not inject enough energy to excite a normal mode of the field. We can put this another way. The resonance frequency of each spring is Ω . As long as the field is not driven at its resonance frequency, part of the driving force does no work because the drive is not in phase with the restoring force of the springs. The motor energy is therefore not transferred efficiently to the spring network. From the dispersion relation $\omega^2 = c^2 k^2 + \Omega^2$, we find:

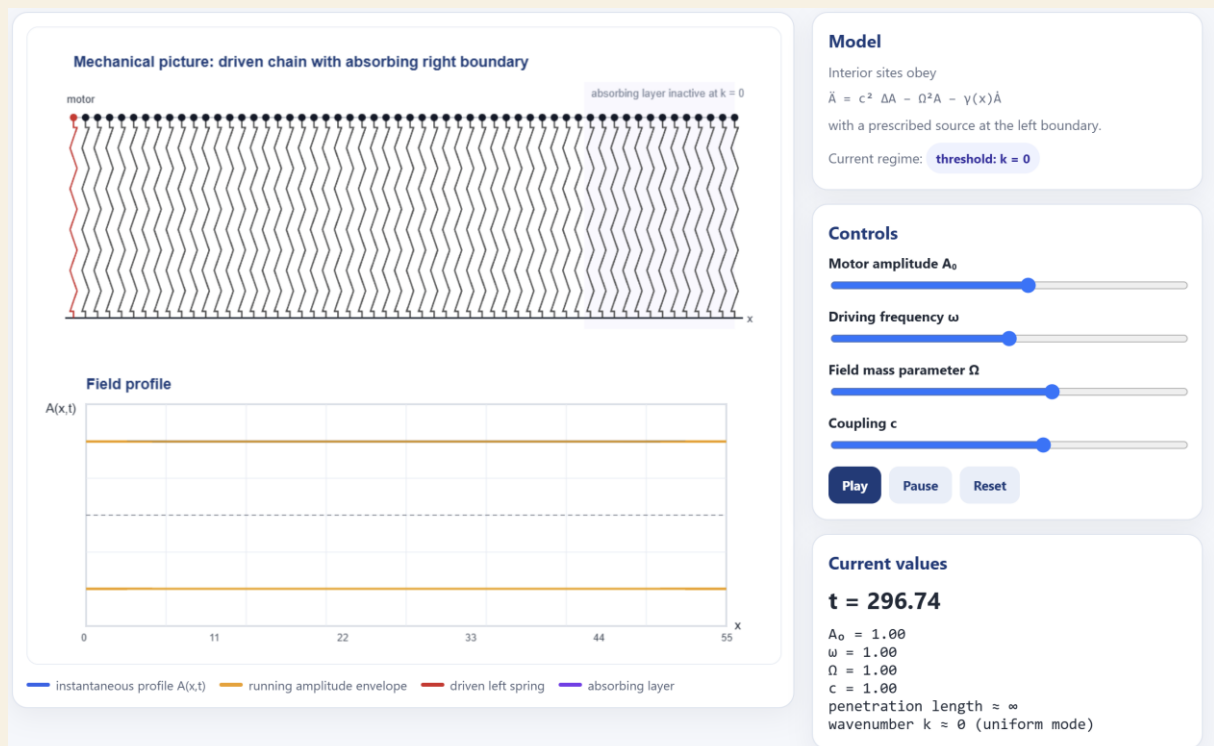
We find that $k=i\sigma$; in other words, the wave number is imaginary!

In the image below, the frequency is only slightly below the mass threshold ($\omega = 0.99 \Omega$), yet this is still not enough for waves to propagate. The field remains "stuck" to the source.



Once $\omega = \Omega$, resonance occurs: the motor drives its spring at its natural frequency. With appropriate boundary conditions, once the steady regime is reached, the spatial part of the solution has the form $\alpha(x) = \alpha_0$, meaning that it does not vary in space. All the springs therefore oscillate in phase. The dispersion relation then gives $k = 0$ and $\omega = \Omega$. We have reached a frequency high enough to create particles at rest!

In the simulation, however, the absorbing layer on the right must be removed in order to reach this regime.



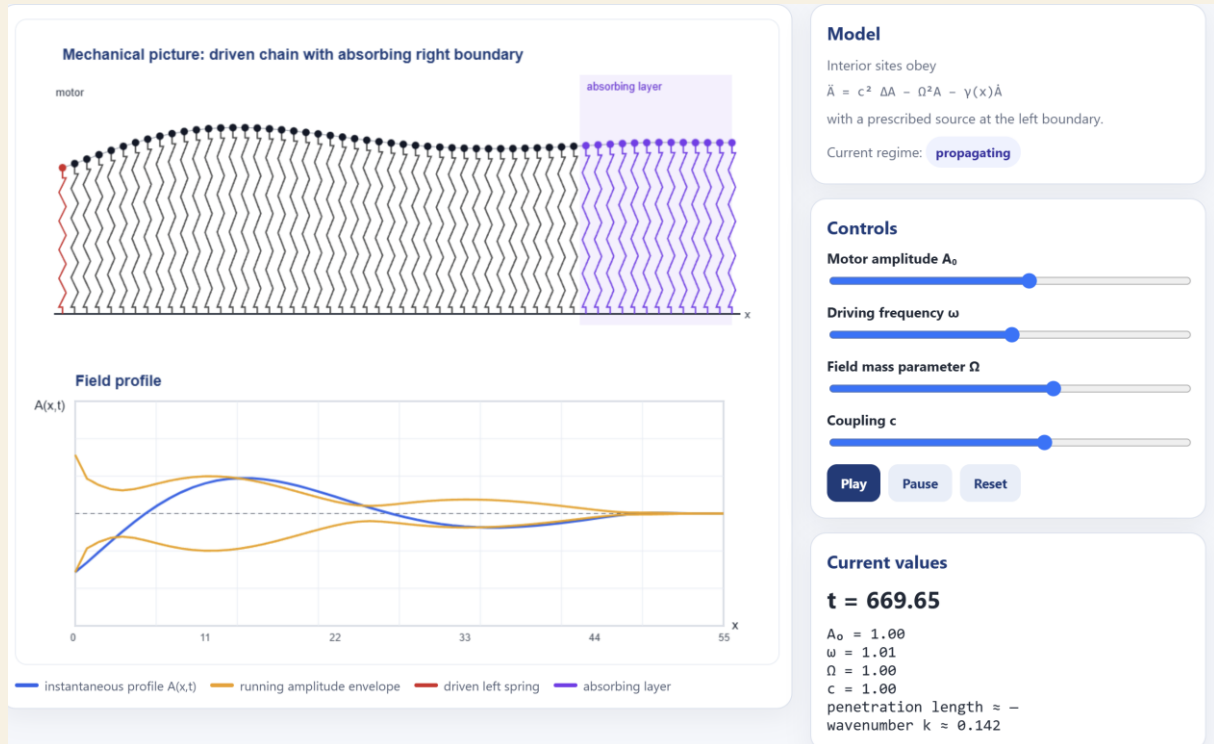
Now, when $\omega > \Omega$, we might think that we have moved away from resonance. But this ignores the force induced by the Laplacian term, which adjusts the effective resonance frequency of the springs so that it matches ω exactly. The spatial part of the solution then becomes:

$$\alpha(x) = \alpha_0 \cos(kx)$$

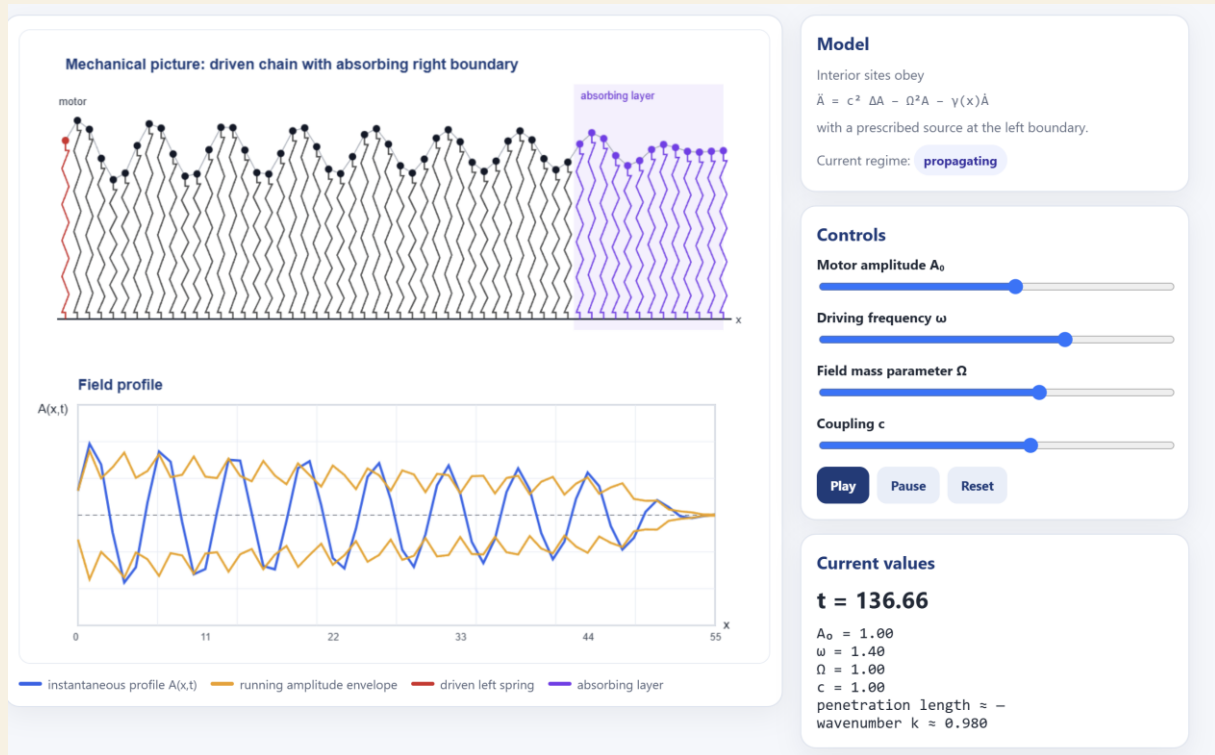
We therefore obtain a travelling wave satisfying the dispersion relation:

$$\omega^2 = c^2 k^2 + \Omega^2$$

Below, $\omega = 1.01 \Omega$



Below, $\omega = 1.4 \Omega$



We can therefore see that the higher the driving frequency, the larger the wave number of the generated waves.

To try the animation, go [to the article presentation page](#).

Particle interpretation

We have already discussed the link between the dispersion relation of waves and the energy relation for a relativistic particle. A wave whose frequency ω satisfies the relation $\omega^2 = c^2 k^2 + \Omega^2$

corresponds to a particle with energy proportional to ω , momentum proportional to k , and mass proportional to Ω . In particle language, the two field regimes correspond to two specific situations:

- If $\omega < \Omega$:

The motor does not supply enough energy to create real particles. As we have seen, the wave number, and hence the momentum, is imaginary. There is then no real propagating mode satisfying the dispersion relation $\omega^2 = c^2 k^2 + \Omega^2$.

The particle is said to be off its "mass shell". This is typically the regime in which the mediator field does not radiate but can still produce an attractive or repulsive effect, often described in popular accounts as the exchange of "virtual particles". But the field interpretation is much less mysterious here. The driving frequency simply has not reached the natural frequency of the field.

- If $\omega \geq \Omega$:

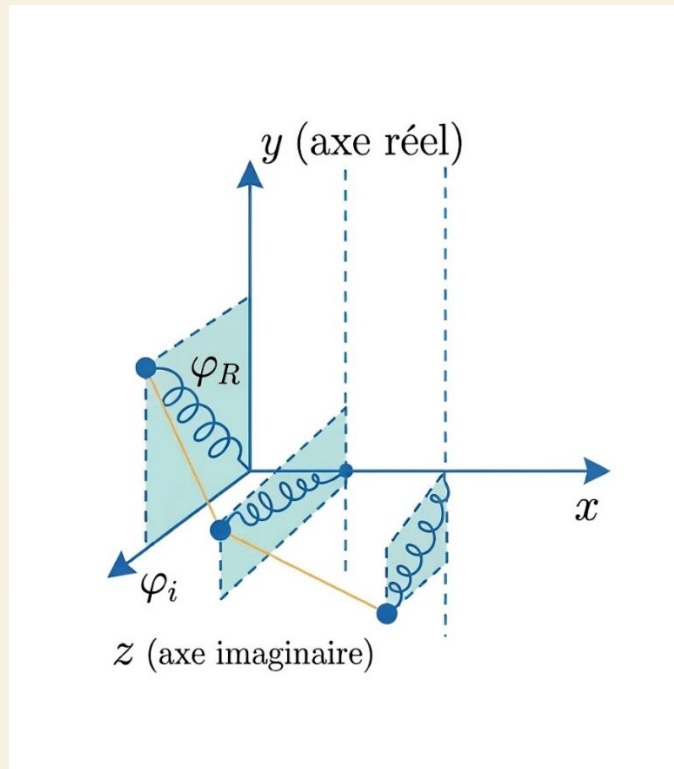
The motor supplies enough energy to create real particles that satisfy the relativistic energy relation. The particles become independent of the source and can propagate freely. To excite the field sustainably, one must therefore drive it at a frequency at least as large as its rest-mass frequency.

II.3 A mechanical model of a charged field

So far we have considered only real mechanical fields. In physics, real fields correspond to uncharged particles, whereas complex fields correspond to charged particles. Here the word charge should be understood generically. It should be identified specifically with electric charge only when the field is coupled to an electromagnetic field.

It is relatively easy to construct an entirely mechanical model of a complex field. We simply introduce one additional degree of freedom for the field.

Consider our spring again. It is fixed at a site, but unlike before it can move in a plane. The spring must be anisotropic, so that the force it exerts on the mass attached to its end depends only on the magnitude of the displacement. The spring displacement therefore has two components: one along the y-axis and one along the z-axis. To distinguish the role of each spatial direction, we align the springs along the x-axis. As before, each small mass attached to its spring is also linked by elastic couplings to its immediate neighbours. We can thus decompose this model into two fields: one corresponding to the displacement along y and the other to the displacement along z. Both fields obey the Klein-Gordon equation. Mathematically, these two field equations are equivalent to a single complex-field equation. The system can therefore be modelled by a complex Klein-Gordon equation, with the real axis corresponding, for example, to the y-axis and the imaginary axis to the z-axis.



The Lagrangian of the complex Klein-Gordon field is:

$$\mathcal{L} = \frac{\partial \phi^*}{\partial t} \frac{\partial \phi}{\partial t} - c^2 \frac{\partial \phi^*}{\partial x} \frac{\partial \phi}{\partial x} - \Omega^2 \phi^* \phi$$

Using the Euler-Lagrange equation, this gives the usual propagation equation for the field:

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \frac{\partial^2 \phi}{\partial x^2} + \Omega^2 \phi = 0$$

The Lagrangian is exactly invariant under a phase transformation by a constant angle α :

$$\begin{aligned}\phi &\rightarrow e^{-i\alpha} \\ \phi^* &\rightarrow e^{i\alpha} \phi^*\end{aligned}$$

This corresponds to rotating the entire wave around the x ,-axis, mixing the real-axis (y) and imaginary-axis (z) components by an angle α without changing the total field amplitude.

According to Noether's theorem, this continuous invariance implies the existence of a conserved four-current (j^μ) satisfying the continuity equation $\frac{\partial j^0}{\partial t} + \frac{\partial j^1}{\partial x} = 0$:

$$\begin{aligned}j^0 &= i \left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t} \right) \\ j^1 &= -ic^2 \left(\phi^* \frac{\partial \phi}{\partial x} - \phi \frac{\partial \phi^*}{\partial x} \right)\end{aligned}$$

Integrating the spatial density of this current over the whole volume gives the conserved total charge (Q):

$$Q = \int j^0 dx = i \int \left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t} \right) dx$$

If we identify what the amplitudes represent in our mechanical model, we find that the charge corresponds to... the angular momentum of the masses around the x-axis. In field physics, the y and z axes form an abstract complex internal space, and the charge of the field corresponds to angular momentum in this abstract space. We will gradually move away from mechanical representations of fields and further into abstraction, because we have reached the point where mechanical analogies increasingly lose their usefulness and begin to introduce unnecessary complexity. But before abandoning the mechanical models, there is something very interesting to examine. There are two types of solutions to the complex equation:

First:

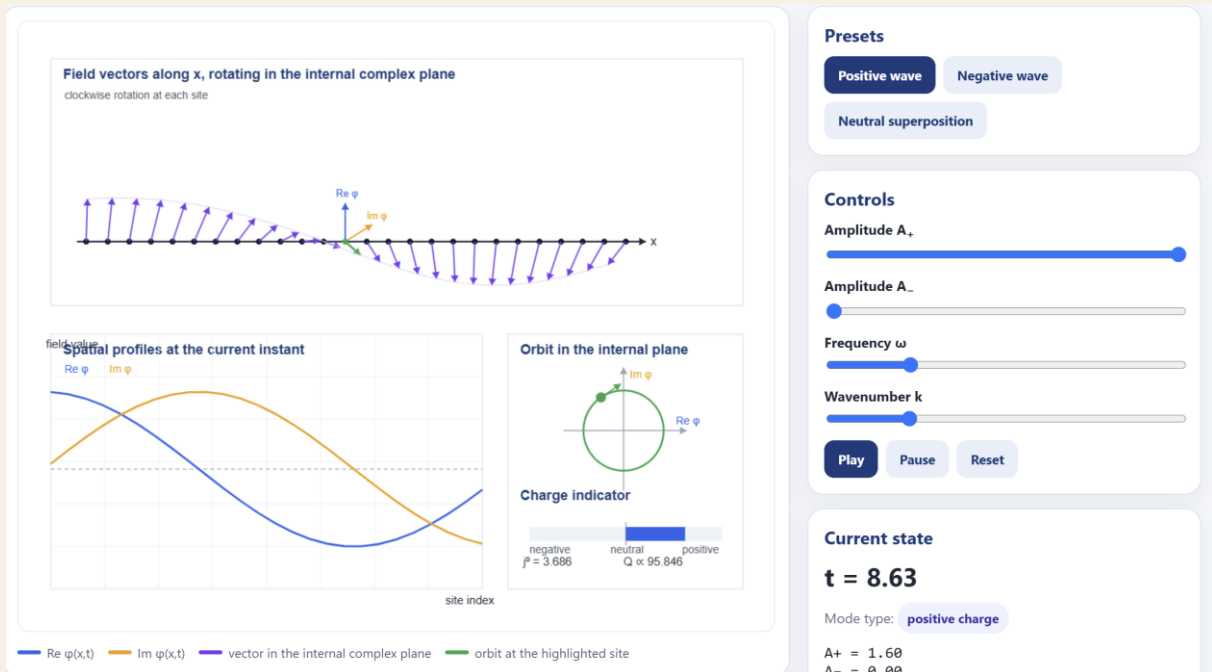
$$\phi_+(x, t) = Ae^{-i(\omega t - k \cdot x)}$$

This solution corresponds to springs rotating at frequency ω clockwise around the centre of their site. Calculating the charge from the charge-density expression j^0 , we obtain:

$$j^0 = 2\omega|A|^2$$

This solution therefore has positive charge.

(See the image below.)



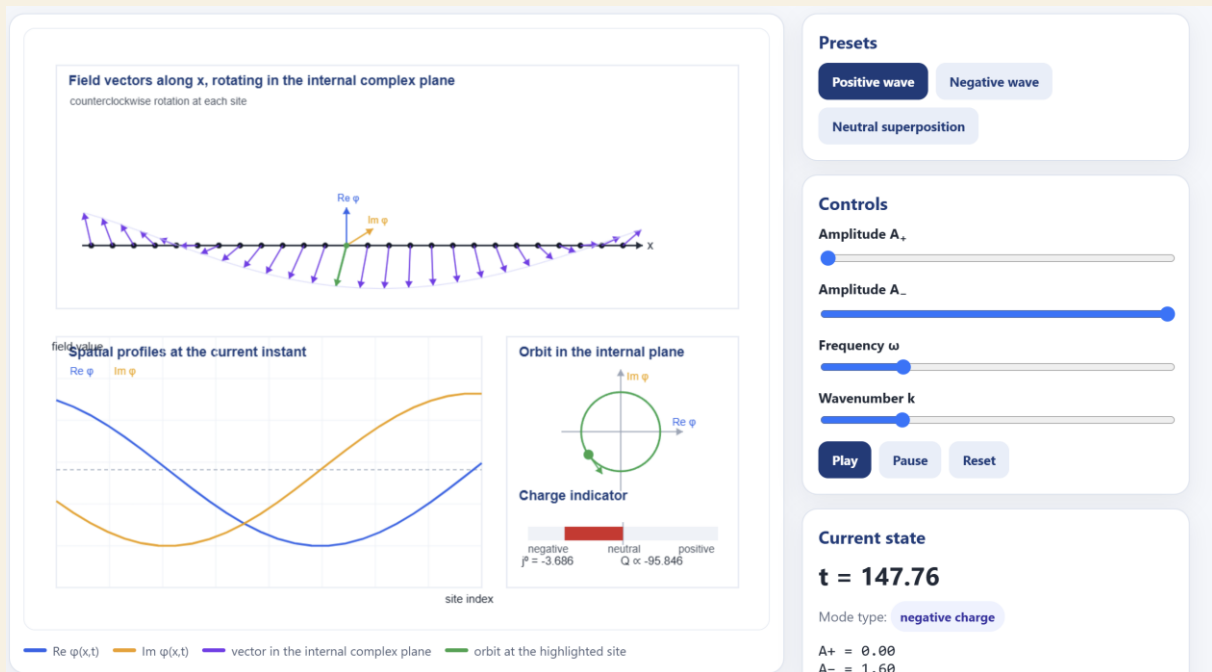
But there are also solutions in which the springs rotate in the opposite direction:

$$\phi_-(x, t) = Ae^{+i(\omega t - k \cdot x)}$$

Their charge density is:

$$j^0 = -2\omega|A|^2$$

This solution therefore has negative charge.



Thus, a complex field has two kinds of excitations carrying opposite charges. In quantum field theory, these are the two kinds of excitations that correspond to matter and antimatter.

We know that one property of matter and antimatter is that they can annihilate. What happens if we superpose a positive-charge solution and a negative-charge solution?

We then have:

$$\phi(x, t) = Ae^{-i(\omega t - \mathbf{k} \cdot \mathbf{x})} + Ae^{+i(\omega t - \mathbf{k} \cdot \mathbf{x})}$$

Using Euler's formula, this simplifies exactly to:

$$\phi = 2A \cos(\omega t - \mathbf{k} \cdot \mathbf{x})$$

In other words, superposing a negatively charged solution and a positively charged solution with the same amplitude and frequency produces a real wave.

We can verify that its charge is zero. The two charges have therefore neutralized each other. Is this annihilation? Not really, because the wave still carries an energy equal to the sum of the energies of the two original waves. It is simply a neutral mode. As long as the field is not coupled to anything else, there is nowhere for that energy to go. But if the field is coupled to another field, the neutral modes can act as periodic sources for it. These sources must, however, drive frequencies at or above the mass threshold. A neutral mode can therefore become the classical analogue of positronium: a bound system made of an electron and a positron. Such systems are very unstable in nature. In the classical case, however, the decay is not complete.

You can create waves with whichever charge you like using [the animation available on the article presentation page](#).

II.4 A first realistic model of interaction between two fields

In what follows, we will bring together the pieces developed so far in order to study realistic models of interacting fields. Behaviours and dynamics that begin to resemble genuine particle physics are now within reach!

The first model we consider is the interaction between a charged Klein-Gordon field coupled to a real Klein-Gordon field. We have already studied interactions among several Klein-Gordon fields in the "sandwich" model of Section II.1. We saw, however, that the coupling in that model was not well suited to producing genuine particle-like dynamics.

The model we examine in this section is much richer. Let us begin by writing down the Lagrangian of the system.

$$\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_\phi + \mathcal{L}_I$$

$$\mathcal{L} = \frac{\partial \psi^*}{\partial t} \frac{\partial \psi}{\partial t} - c^2 \frac{\partial \psi^*}{\partial x} \frac{\partial \psi}{\partial x} - \Omega_\psi^2 \psi^* \psi + \frac{1}{2} \left[\left(\frac{\partial \phi}{\partial t} \right)^2 - c^2 \left(\frac{\partial \phi}{\partial x} \right)^2 - \Omega_\phi^2 \phi^2 \right] - g \psi^* \psi \phi$$

Here is the role of each term:

- **The complex "matter" field $\mathcal{L}_\psi$:** this is the Klein-Gordon dynamics of a complex scalar field of mass Ω_ψ . We have just studied the behaviour of such a field in the previous section.
- **The real "mediator" field $\mathcal{L}_\phi$:** this is the Klein-Gordon dynamics of a real scalar field of mass Ω_ϕ . We have already studied such a field.
- **The interaction term $\mathcal{L}_I$:** this is the coupling term. The minus sign is a standard physical convention: the term acts like an interaction potential energy. The constant g is the coupling constant. We have not yet studied a coupling of this form, so it is worth taking some time to spell out its physical meaning. This coupling plays the role of an interaction potential energy, but it is no longer a "spring-to-spring" coupling. It is linear in ϕ and

quadratic in ψ . Notice also that the two fields play asymmetric roles in the coupling. They therefore do not act on one another in exactly the same way.

Let us derive the Euler-Lagrange equations for these two fields. The fields ψ and ψ^* are not independent: because they are complex conjugates, one determines the other. The true degrees of freedom are the real and imaginary parts. Nevertheless, one can greatly simplify the calculation while preserving the same physics by treating ψ and ψ^* as independent fields and deriving an Euler-Lagrange equation for each. The corresponding field equations are therefore:

For the field ψ :

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial x^2} + \Omega_\psi^2 \psi = -g\phi\psi$$

For the field ψ^* :

$$\frac{\partial^2 \psi^*}{\partial t^2} - c^2 \frac{\partial^2 \psi^*}{\partial x^2} + \Omega_\psi^2 \psi^* = -g\phi\psi^*$$

For the field ϕ :

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \frac{\partial^2 \phi}{\partial x^2} + \Omega_\phi^2 \phi = -g\psi^*\psi$$

Let us analyse these equations. First:

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial x^2} + \Omega_\psi^2 \psi = -g\phi\psi$$

can be rearranged as:

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial x^2} + (\Omega_\psi^2 + g\phi)\psi = 0$$

This has exactly the form of the free Klein-Gordon equation for a field with a variable mass term: $\Omega_{\text{eff}}^2 = \Omega_\psi^2 + g\phi$.

This means that when ϕ is positive, it strengthens the restoring term Ω_ψ^2 and acts like a "force" on ψ that tends to prevent it from reaching large amplitudes. It also increases the oscillation frequency and raises the resonance threshold: the field ψ is therefore harder to excite in a region where ϕ is large. But ϕ can also be negative. In that case, it makes the field ψ less stiff. Something remarkable then happens once

$$\Omega_\psi^2 + g\phi < 0.$$

In that case, the effective mass Ω_{eff} becomes imaginary! The "restoring force" now pulls in the same direction as ψ and, instead of making the field oscillate about its equilibrium value, tends to amplify it without bound. This is called a tachyonic regime. Some fields may have undergone such a regime during cosmic inflation in the early Universe. In the real world, because fields are coupled to other fields, such a regime does not last indefinitely: the fields involved rapidly transfer their energy to other fields.

Now let us insert a solution of the form:

$$\psi(t, x) = R(t, x)e^{iS(t, x)}$$

Separating the real and imaginary parts and rearranging the equations slightly, we obtain for the imaginary part:

$$\frac{\partial}{\partial t} \left(R^2 \frac{\partial S}{\partial t} \right) - c^2 \frac{\partial}{\partial x} \left(R^2 \frac{\partial S}{\partial x} \right) = 0$$

We will return shortly to the physical meaning of this equation. For the real part, we obtain:

$$\left(\frac{\partial S}{\partial t} \right)^2 - c^2 \left(\frac{\partial S}{\partial x} \right)^2 = \Omega_\psi^2 + g\phi + \frac{1}{R} \left(\frac{\partial^2 R}{\partial t^2} - c^2 \frac{\partial^2 R}{\partial x^2} \right)$$

This equation has the form of a relativistic Hamilton-Jacobi equation. If we set $E = -\frac{\partial S}{\partial t}$ and $p = \frac{\partial S}{\partial x}$, the equation becomes $E^2 = c^2 p^2 + \Omega_\psi^2 + g\phi + Q$, with $Q = \frac{1}{R} \left(\frac{\partial^2 R}{\partial t^2} - c^2 \frac{\partial^2 R}{\partial x^2} \right)$. Although we are still working entirely within a classical field framework, the term Q is sometimes called the "quantum potential", by reference to the de Broglie-Bohm pilot-wave interpretation of quantum mechanics. When this term remains sufficiently small, the Hamilton-Jacobi equation therefore gives us an approximate picture of the dynamics of a wave packet. Like a particle attracted toward lower potential energy, our wave packet experiences an effective force toward lower values of ϕ .

Taking the gradient of this Hamilton-Jacobi equation gives the analogue of the Euler equation of fluid mechanics. Setting $v = \frac{c^2 p}{E}$, we obtain:

$$\frac{\partial p}{\partial t} + v \frac{\partial p}{\partial x} = -\frac{1}{2E} \frac{\partial}{\partial x} (g\phi + Q)$$

which tells us that each infinitesimal element of the wave experiences a force equal to $-\frac{1}{2E} \frac{\partial}{\partial x} (g\phi + Q)$.

Let us return to our equation $\frac{\partial}{\partial t} \left(R^2 \frac{\partial S}{\partial t} \right) - c^2 \frac{\partial}{\partial x} \left(R^2 \frac{\partial S}{\partial x} \right) = 0$.

Using the substitutions $E = -\frac{\partial S}{\partial t}$, $p = \frac{\partial S}{\partial x}$, $v = \frac{c^2 p}{E}$ and $\rho = ER^2$, we obtain:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v) = 0$$

This is the continuity equation, which is in fact the charge-conservation equation. ρ is proportional to the charge density (see Noether's theorem above).

In summary, inserting our solution into the field equation yields two equations: the real part governs the dynamics of the wave packet, while the imaginary part gives the equation of charge conservation.

We now turn to the equation for the field ϕ .

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \frac{\partial^2 \phi}{\partial x^2} + \Omega_\phi^2 \phi = -g\psi^* \psi$$

For the moment, suppose that ψ is constant and zero everywhere except within a very small region, and that the system is in equilibrium ($\frac{\partial^2 \phi}{\partial t^2} = 0$). This is exactly the case we studied with the spring model, when we pulled a single spring with a fixed force and examined the overall shape of the spring field. We found that $A(x) = Be^{-\frac{\Omega}{c}|x|}$.

The difference is that here, because of the sign of the coupling, the field ψ makes the field ϕ "downward", that is, toward negative values of ϕ . Apart from this sign difference, applying the same reasoning as before gives:

$$\phi(x) = -\phi_0 e^{-\frac{\Omega\phi}{c}|x|}$$

where ϕ_0 is a positive real number that depends on the amplitude of the source ψ . In other words, ψ "digs a well" in the field $\phi(x)$, since $\psi^*\psi > 0$ (the modulus of a complex number is always positive).

Now let us look again at our Euler equation:

$$\frac{\partial p}{\partial t} + v \frac{\partial p}{\partial x} = -\frac{1}{2E} \frac{\partial}{\partial x} (g\phi + Q)$$

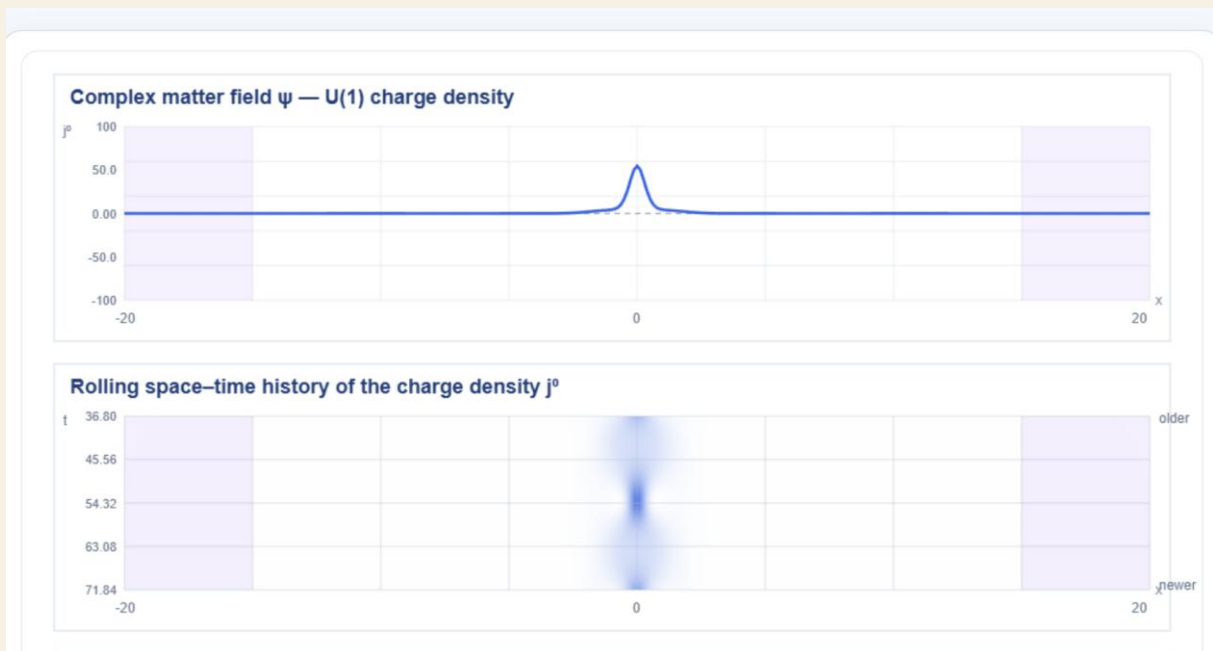
If we consider a situation close to the static case (a stationary wave packet), we have:

$$\frac{\partial p}{\partial t} + v \frac{\partial p}{\partial x} \approx 0$$

And therefore:

$$\frac{\partial}{\partial x} \left(g\phi_0 e^{-\frac{\Omega\phi}{c}|x|} \right) \approx -\frac{\partial Q}{\partial x}$$

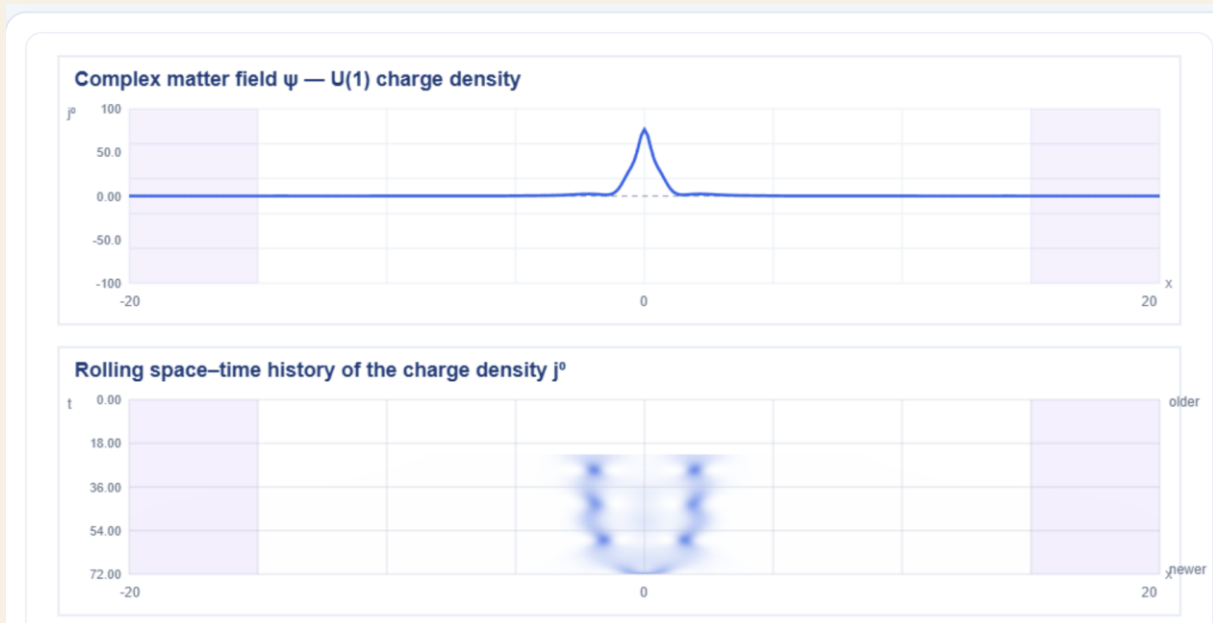
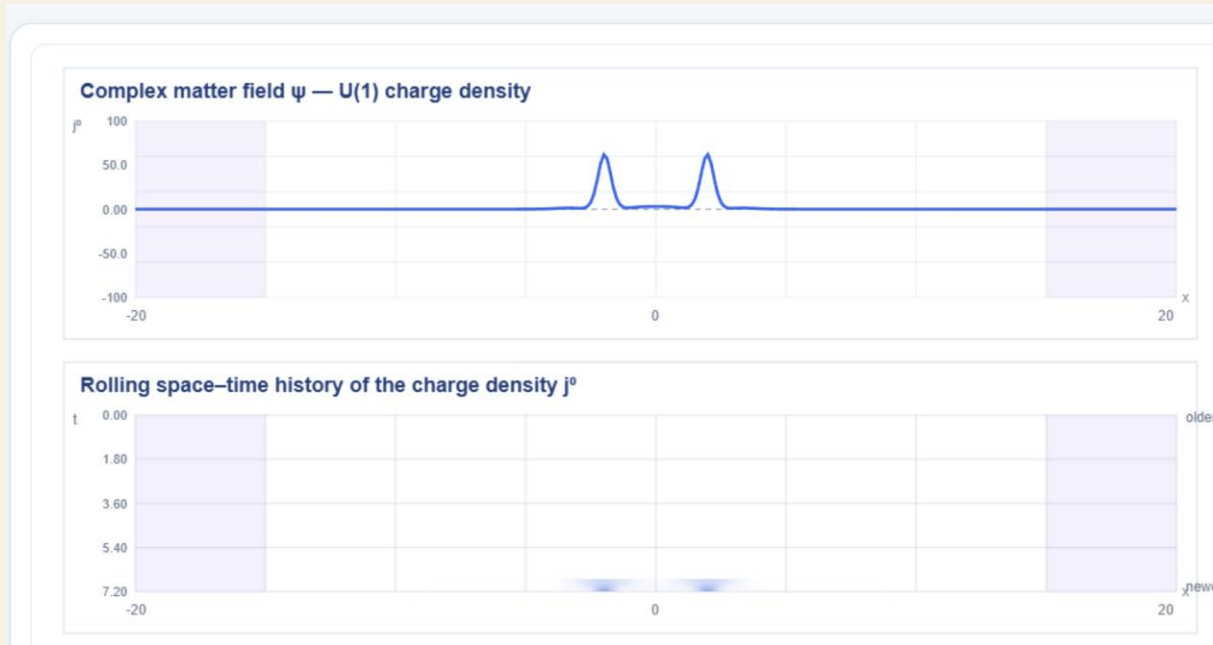
This equation describes a balance between an attractive force associated with the potential of the field ϕ and a repulsive contribution from the quantum potential, which acts like an internal pressure in the field ψ . As a result, wave packets tend to stabilize into structures whose width depends, among other things, on the mass of the field $\phi(x)$. Without going into the details, because the two fields feed back on one another, if the wave packet becomes too broad the attractive force dominates and tends to squeeze it; if it becomes too narrow, the pressure takes over and the packet expands again. The wave packet therefore appears to "breathe". Such a packet is called a soliton, and it is the closest thing to a genuinely stable particle that we obtain in classical field theory.

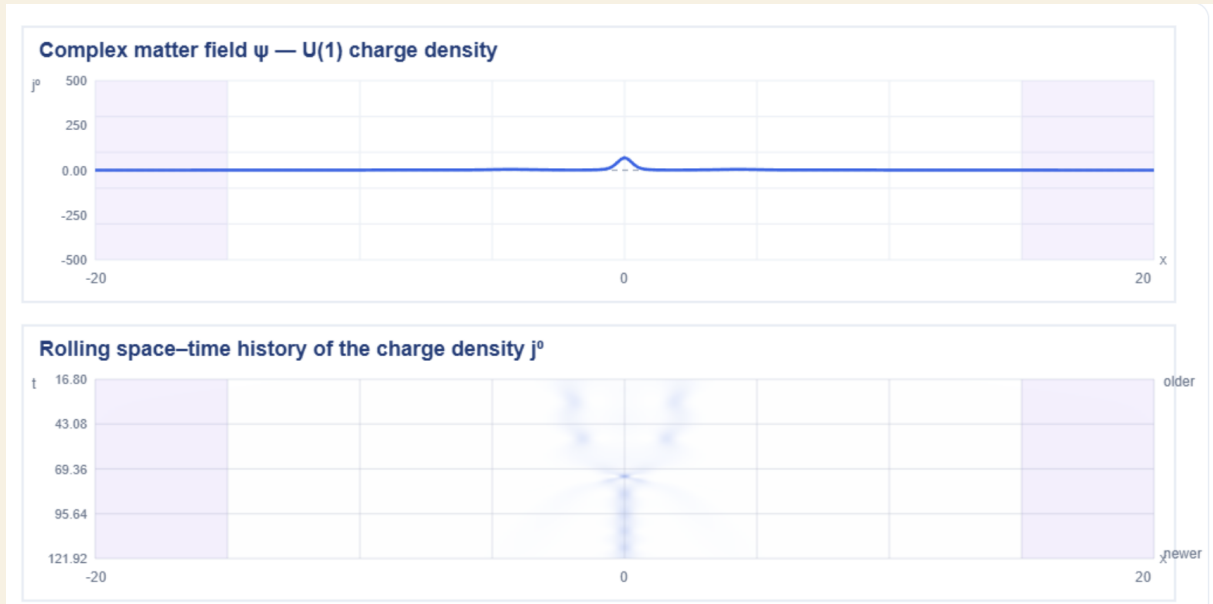


In this animation, the upper panel shows the charge distribution at a given instant. The lower panel shows its evolution over time: the charge spreads out and contracts again at a regular rhythm. This is

the characteristic breathing of the soliton. You can experiment with your own soliton using [the animation available on the article presentation page](#). Do not set the coupling constant too high, or it is GAME OVER: the tachyonic regime! (The animation then stops.)

Moreover, two solitons exert an effective force on one another proportional to $\frac{\partial}{\partial x} \left(g \phi_0 e^{-\frac{\Omega \phi}{c} r} \right)$, where r is the distance between them. What happens when two solitons collide is particularly interesting, and this time the outcome depends on their charge. If the two solitons carry the same charge, charge conservation prevents the two waves from cancelling completely. If their initial kinetic energy is lower than their interaction potential energy, the two solitons merge.





A warning about these three images, which capture three stages in the merger of two solitons: the last image is misleading because it can make the final soliton look tiny compared with the two initial solitons. The reason is simply that the scale has changed. The final soliton is therefore not smaller, but in fact much larger.

The newly formed soliton then sheds the kinetic energy of the two parent solitons as radiation in ϕ , provided that the collision is violent enough to excite modes above ϕ its mass threshold. Indeed, in that case, starting from the equation

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \frac{\partial^2 \phi}{\partial x^2} + \Omega_\phi^2 \phi = -g\psi^*\psi$$

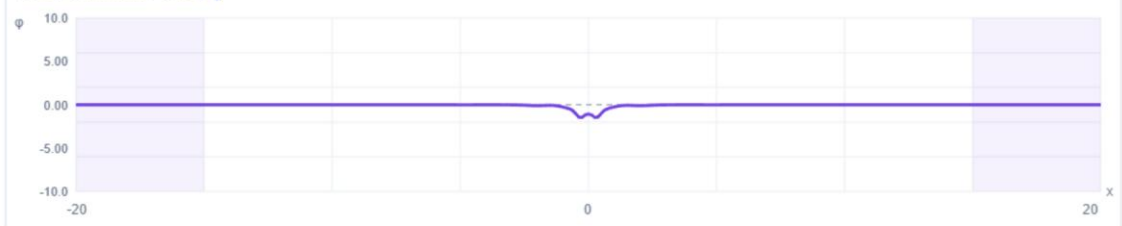
if $\psi^*\psi$ changes rapidly, the situation is equivalent to a network of springs being shaken violently at many different frequencies. Frequencies greater than or equal to Ω_ϕ resonate with the field ϕ and therefore produce radiation.



Rolling space-time history of the charge density j^0

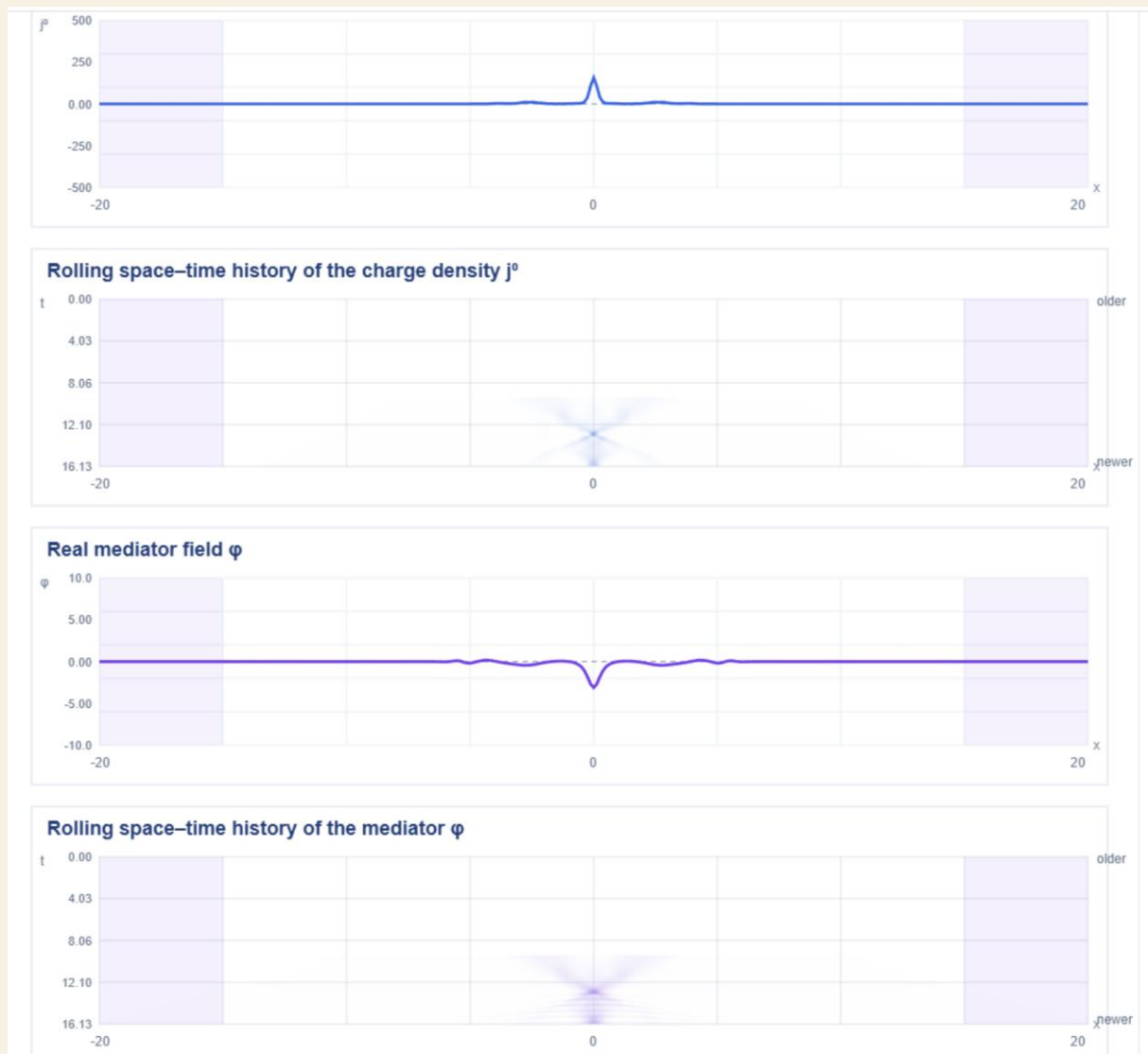


Real mediator field ϕ



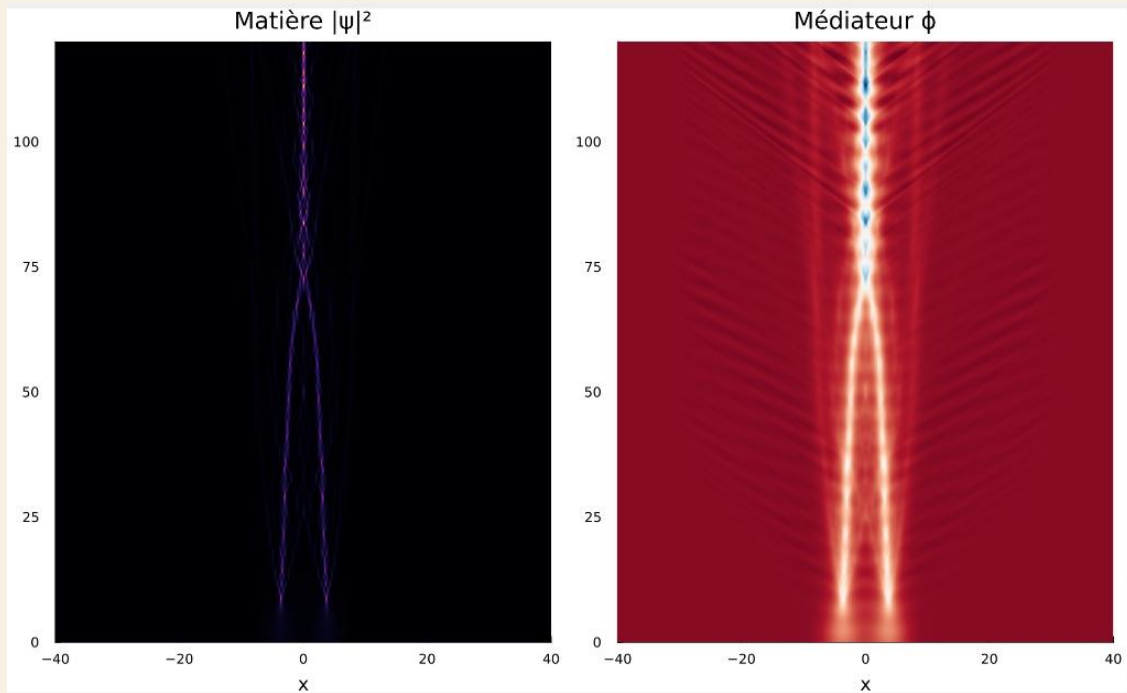
Rolling space-time history of the mediator ϕ





Below, panels 3 and 4 show the emission of waves in the mediator field after the two solitons have merged.

And below is a higher-resolution Julia simulation of the same merger process for two solitons carrying the same charge. Time now runs from bottom to top.



The left panel represents the density of the matter field ($|\psi|^2$), while the right panel shows the structure of the scalar mediator field (ϕ).

The radiation emitted by the merger and the breathing of the final soliton are clearly visible here.

The most interesting behaviour appears when two solitons of opposite charge collide. Under suitable conditions, destructive interference can occur.

For example, suppose we superpose two waves of equal amplitude:

Then $\psi_{total} = 2R \cos(px - \omega t)$; the field becomes real and the charges neutralize. However, the source of the field ϕ is not the charge but $\psi^*\psi$.

Now $\psi^*\psi = 4R^2 \cos^2(px - \omega t)$

The field ψ can therefore act as a periodic oscillator that generates radiation in ϕ . Since energy is conserved, the radiating source is damped and eventually loses its energy. This would amount to a relatively "gentle" matter-antimatter annihilation. But the actual dynamics can be more violent: at the node where the two trajectories meet, strong gradients in the field ψ can increase the pressure associated with the quantum potential, and the soliton can break apart violently, emitting energetic ϕ radiation in the process. The price of this violence is that part of the energy in the field ψ survives as a kind of "dust", so the annihilation is not complete. Classical field theory can therefore reproduce matter-antimatter annihilation processes, as quantum field theory does. Unlike the quantum case, however, these processes are not complete: matter and antimatter do not disappear entirely in a flash of radiation, only partially. Nor is the process automatic: the collision must excite frequencies in the field ϕ above its mass threshold. The solitons must therefore be concentrated enough to produce large gradients, while remaining stable enough to avoid the tachyonic regime, which prevents them from being too concentrated. The parameter window for obtaining a clean disintegration is therefore rather

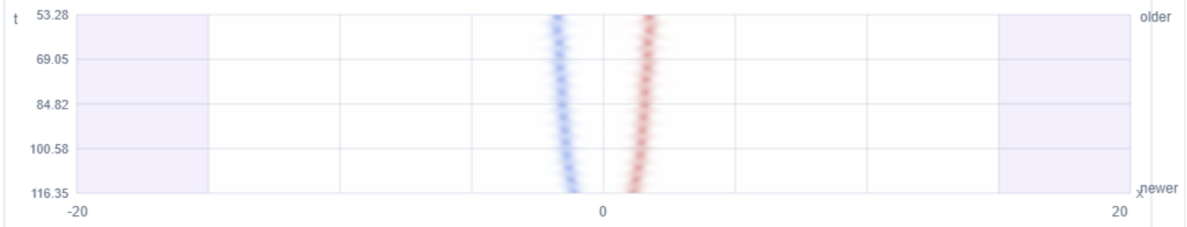
narrow. Even so, it is already a striking achievement of the model that it can reproduce phenomena analogous to antimatter and matter-antimatter annihilation.



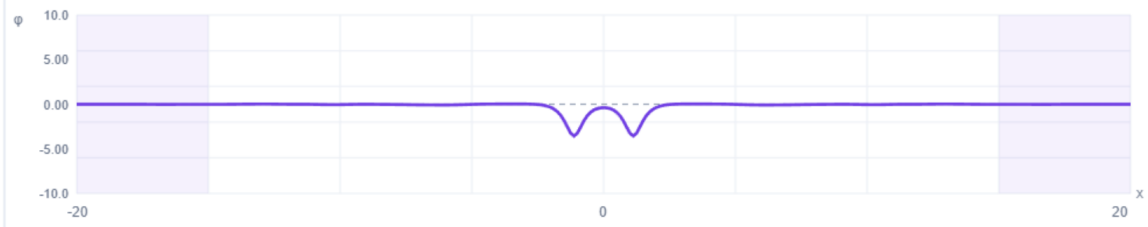
Complex matter field ψ — U(1) charge density



Rolling space-time history of the charge density j^0



Real mediator field ϕ



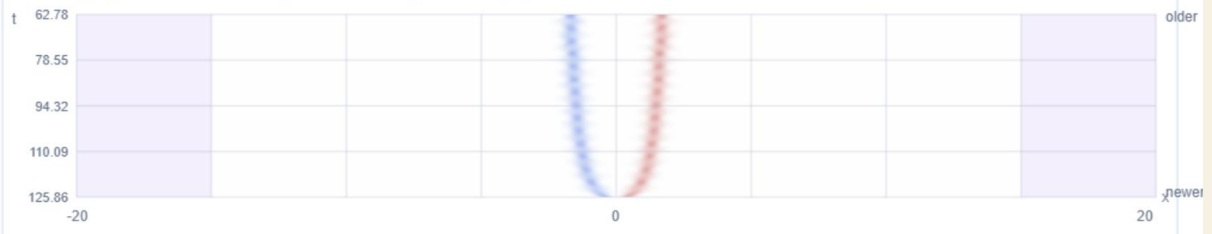
Rolling space-time history of the mediator ϕ



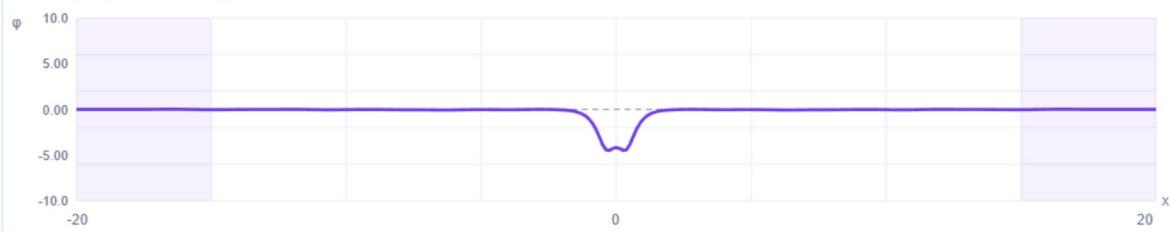
Complex matter field ψ — U(1) charge density



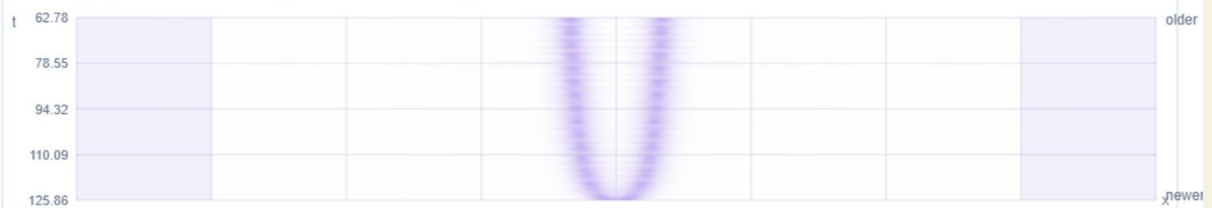
Rolling space–time history of the charge density j^0



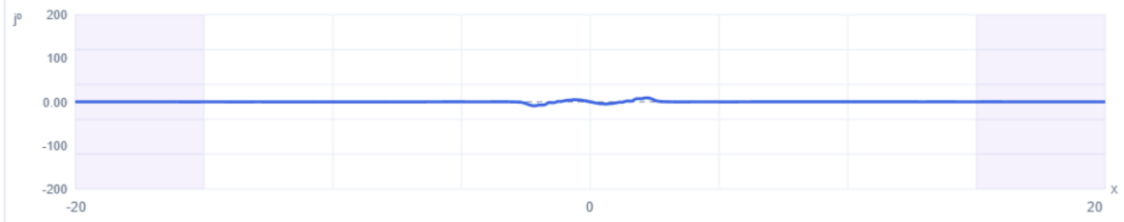
Real mediator field ϕ



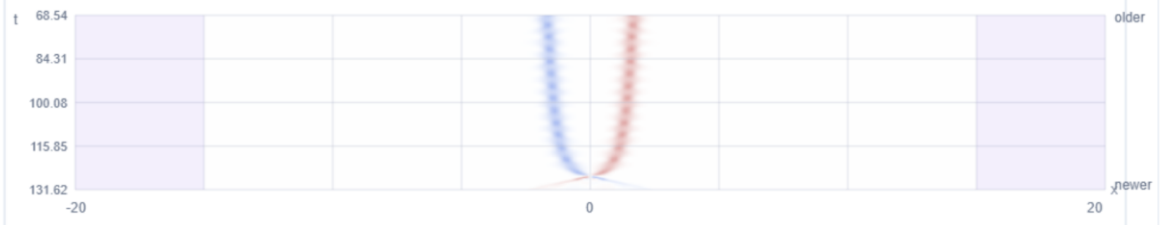
Rolling space–time history of the mediator ϕ



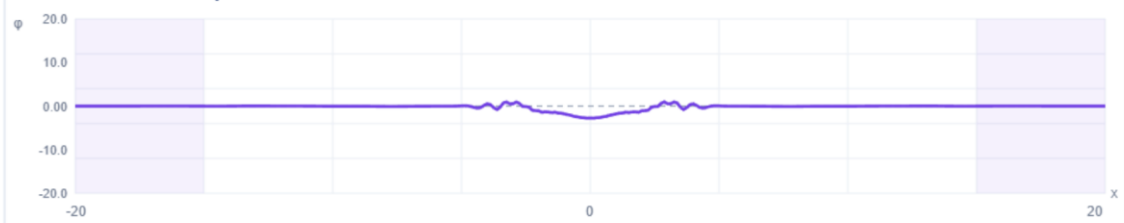
Complex matter field ψ — U(1) charge density j^0



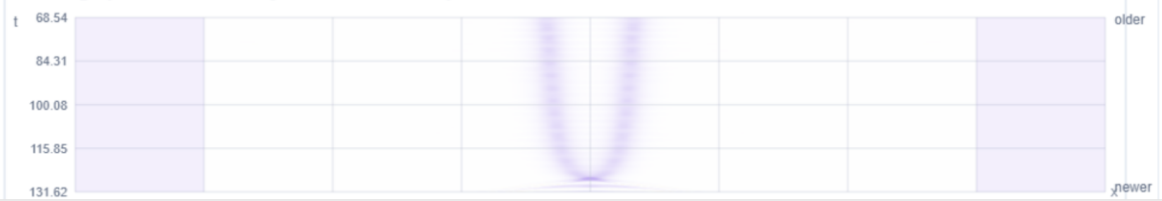
Rolling space-time history of the charge density j^0



Real mediator field ϕ



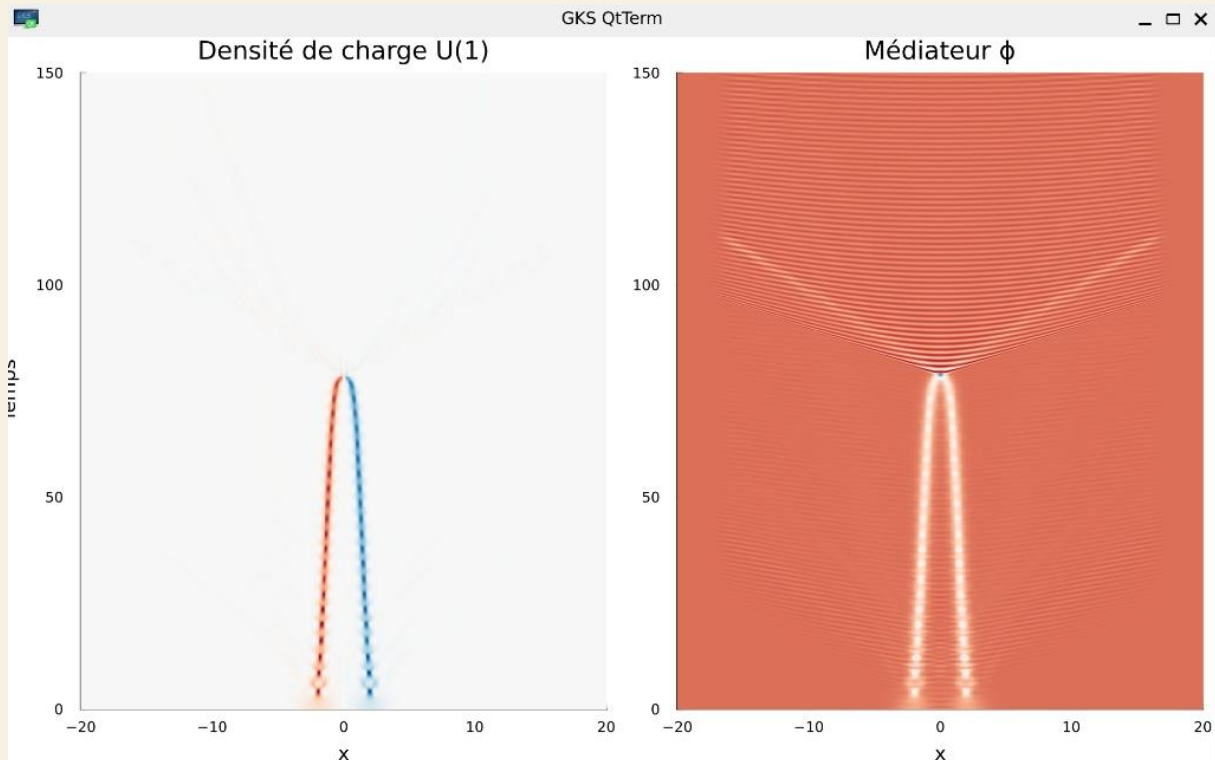
Rolling space-time history of the mediator ϕ



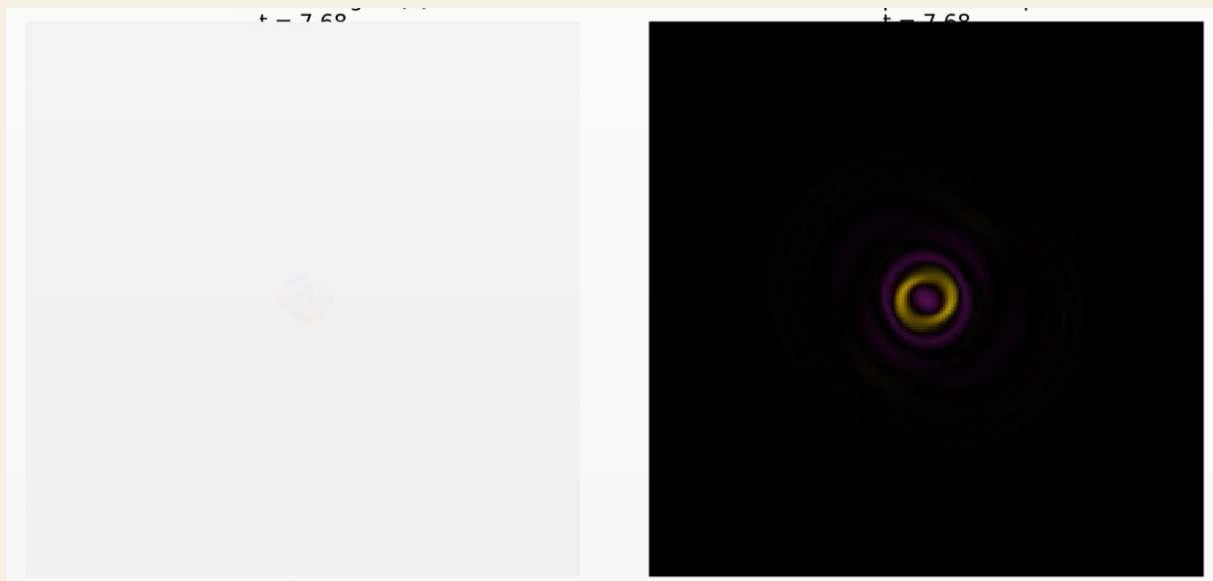


Try your own matter-antimatter collisions in [the same animation, available on the presentation page](#). Be warned: obtaining a clear annihilation is actually difficult. The window between solitons that are too concentrated (tachyonic regime) and not concentrated enough (a process too gentle to excite the field) is extremely narrow.

Below is an image of the same process from a higher-resolution Julia simulation:



On [the article presentation page](#), I also provide a video and an animation of a two-dimensional collision followed by annihilation. The image below shows a post-annihilation stage.



One final phenomenon that this model can account for is the creation of matter and antimatter from radiation. Let us return to the coupled evolution equation for the matter field ψ in the presence of the scalar mediator field ϕ :

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial x^2} + \Omega_\psi^2 \psi = -g\phi\psi$$

This fundamental equation contains the complete mechanism for creating particle-antiparticle pairs. However, a closer look at the source term on the right-hand side, $(-g\phi\psi)$, reveals an important physical constraint: this term is proportional to ψ . If space is initially empty of matter, $(\psi = 0)$, the

source term is exactly zero. A scalar wave ϕ , no matter how energetic, passing through a perfectly empty classical vacuum will produce nothing.

For matter creation to begin, some initial "seed" must be present. In quantum field theory, such a seed is naturally supplied by quantum vacuum fluctuations. Even in the ground state of a quantum field ψ , its lowest-energy state, the mean-square fluctuation of the field $\langle \psi^2 - \langle \psi \rangle^2 \rangle$ is not zero. In other words, a measurement of the field in its ground state need not return zero. This is what quantum vacuum fluctuations mean. They are often pictured as the field constantly oscillating, or as pairs of virtual particles continually appearing and annihilating. These images are misleading: the quantum state of the field itself can be perfectly stationary; it simply is not a state with a definite field value.

In classical physics, there is no exact counterpart of this vacuum state. In our model, the lowest-energy field configuration is simply one in which the field is zero everywhere and at all times. Nevertheless, we can mimic the role of the quantum vacuum by adding tiny fluctuations to the otherwise empty classical field. This is enough to allow the field ψ to drive those fluctuations toward high energies, providing a classical analogue of particle-antiparticle pair creation.

To understand how the wave ϕ interacts with these small fluctuations to create macroscopic matter, we must consider the dynamics of complex waves. In field theory, as we have seen, the direction of temporal phase evolution distinguishes matter from antimatter:

- A matter mode (particle) with wave vector k and energy ω_k is written:

$$\psi_k \propto e^{i(kx - \omega_k t)}$$

- An antimatter mode (antiparticle) with wave vector P and energy ω_p is written with the opposite temporal phase:

Because the mediator field ϕ is real, a travelling wave of momentum q and frequency ω_q decomposes into the sum of two complex-conjugate waves:

$$\phi \propto e^{i(qx - \omega_\phi t)} + e^{-i(qx - \omega_\phi t)}.$$

When an exciting wave ϕ sweeps through space, it interacts with a matter fluctuation ψ_k . The product of these waves in the coupling term $(-g\phi\psi_k)$ generates new waves through spatiotemporal beating.

$$g\phi\psi_k \propto (e^{i(qx - \omega_\phi t)} + e^{-i(qx - \omega_\phi t)}) \times e^{i(kx - \omega_k t)} = e^{i((k+q)x - (\omega_\phi + \omega_k)t)} + e^{i((k-q)x - (\omega_k - \omega_\phi)t)}$$

The term $e^{i((k-q)x - (\omega_k - \omega_\phi)t)}$ is of particular interest here. If $k - q = -P$ and $\omega_\phi - \omega_k = \omega_p$, then this driving term resonates with the antimatter mode $\psi_p \propto e^{-i(Px - \omega_p t)}$. It therefore strongly amplifies that mode and creates antimatter. A symmetric process occurs when the wave ϕ interacts with ψ_p . This time:

$$g\phi\psi_p \propto (e^{i(qx - \omega_\phi t)} + e^{-i(qx - \omega_\phi t)}) \times e^{-i(Px - \omega_p t)} = e^{i((q-P)x - (\omega_\phi - \omega_p)t)} + e^{-i((p+q)x - (\omega_p + \omega_\phi)t)}$$

The term $e^{i((q-P)x - (\omega_\phi - \omega_p)t)}$ then provides a drive that resonates with the wave ψ_k provided that $q - P = k$ and $\omega_\phi - \omega_p = \omega_k$, exactly the same conditions as before. A feedback process also develops: as ψ_p grows, the drive resonant with ψ_k becomes stronger, and as ψ_k grows, the drive resonant with ψ_p becomes stronger, until the wave ϕ has transferred away its energy. Because charge is conserved, the two processes must of course occur simultaneously. This phenomenon is called

parametric resonance, and it may have played an important role at the end of cosmic inflation during the phase known as preheating. The field responsible for inflation, the inflaton, was then highly excited and could enter parametric resonance with other massive fields, producing a cascade of particles and antiparticles.

If we translate all this into particle language, interpreting the p, q et k terms as momenta and the ω_p, ω_k and ω_ϕ terms as energies, the resonance conditions directly reproduce conservation of momentum and energy:

$$\omega_\phi = \omega_k + \omega_p \text{ et } q = k + P$$

for a decay process in which a particle of energy ω_ϕ and momentum q produces a particle-antiparticle pair with energies ω_k and ω_p and momentum k et P .

Finally, the resonance conditions impose a strict threshold for the process. Since the minimum frequency of a massive field is set by its mass ($\omega \geq \Omega_0$), the incident radiation ϕ must have a frequency at least twice the mass of ψ :

$$\omega_\phi \geq 2\Omega_\psi$$

If the energy of the field ϕ exceeds this threshold, the excess is converted directly into kinetic energy of the daughter particles, which move apart with opposite momenta.

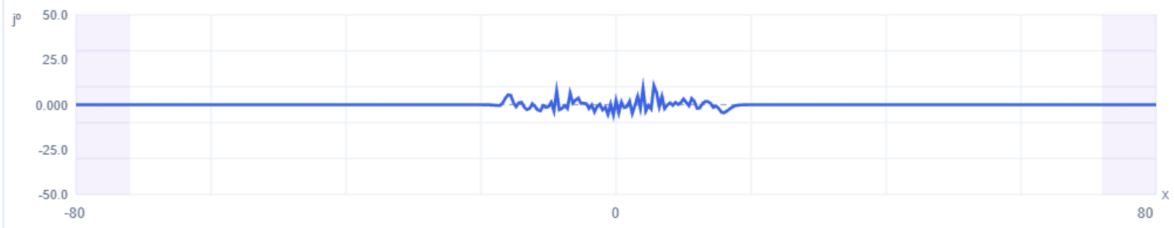


In this image, the charged complex field is almost zero. The visible fluctuations are actually extremely small (see the scale in the upper-left corner). The mediator field is strongly excited at the centre.

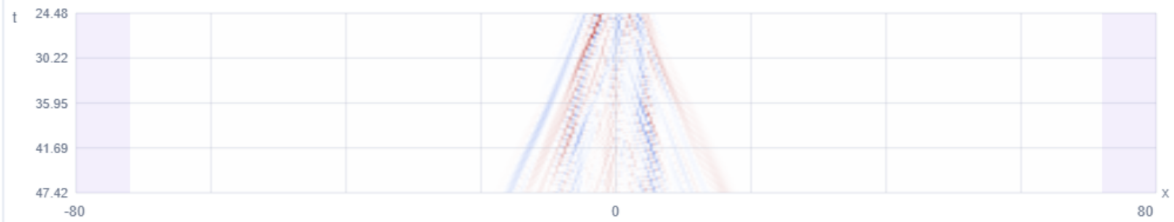


The creation of "charged matter" begins here: the mediator field excites modes of the charged field.

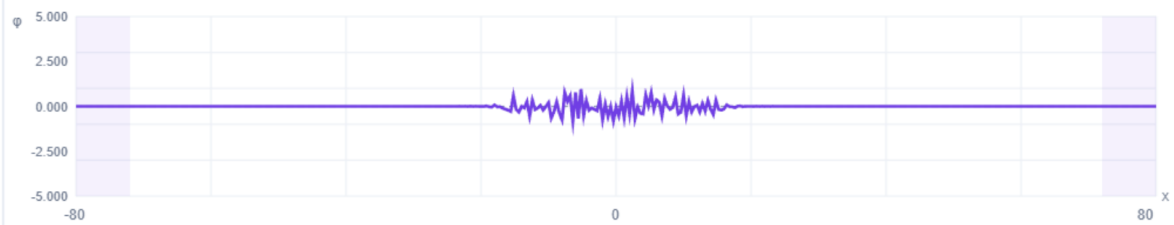
Complex matter field ψ — U(1) charge density



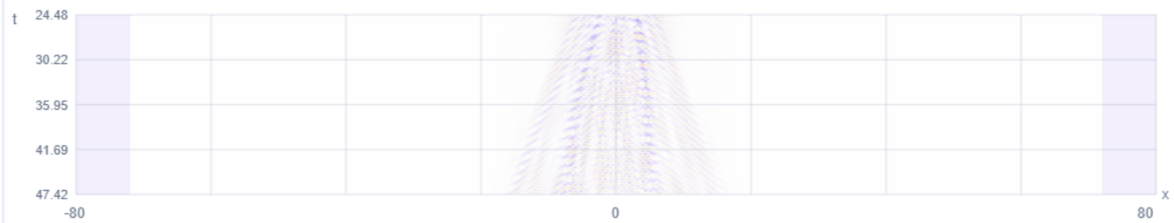
Rolling space-time history of the charge density j^0



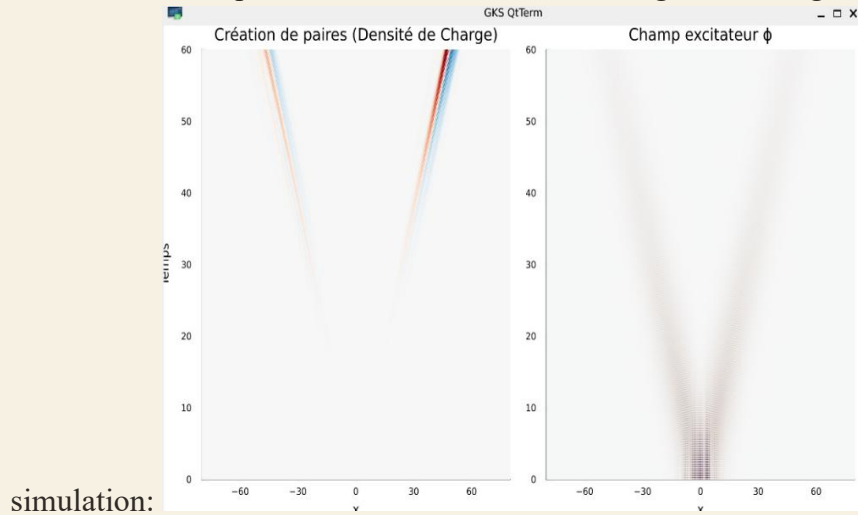
Excitator / mediator field ϕ



Rolling space-time history of the pump field ϕ



A "shower" of soliton-antisoliton pairs is emitted. Below is an image from a higher-resolution Julia



God stirred the field ϕ , and charged matter came to be. One of the best images for a creator of matter would be a guitarist setting strings into vibration: matter would be the sound produced by the guitar!



*Figure 1: The Creator stirs the scalar mediator field; as it loses its excitation, excitations of the charged field are born.
Better than stardust: you are notes of music!*



Figure 2: For those who are more convinced by simulation theory...



Figure 3: Finally, the Pastafarian version of the creation of matter.

Back to the "emergence of relativity"

Playtime is over.

In Part 3, using a very simple mechanical field model, we explored the relativistic behaviour that emerged from wave-packet dynamics. In particular, we saw that the wave phase at the centre of a packet behaved like a kind of clock undergoing time dilation (the famous de Broglie clock). At that stage, however, we could not illustrate length contraction because we lacked a stable structure.

With the present model, we can now demonstrate even more clearly relativistic effects that emerge naturally from the dynamics of solitons.

To do this, we will carry out a particle-accelerator experiment! We prepare a stable soliton at rest, with only a very weak breathing motion. We measure its breathing period and its average width. Then, after some time, we switch on an accelerating electric field. In the previous chapter, we already introduced a mechanical model of a Klein-Gordon field coupled to an electromagnetic field. We simply need to add an interaction term to the dynamical equation of the charged field:

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2} - (m_\psi^2 + g\phi)\psi$$

$$\rightarrow \frac{\partial^2 \psi}{\partial t^2} = \left(\frac{\partial}{\partial x} - ieA_x(t) \right)^2 \psi - (m_\psi^2 + g\phi)\psi$$

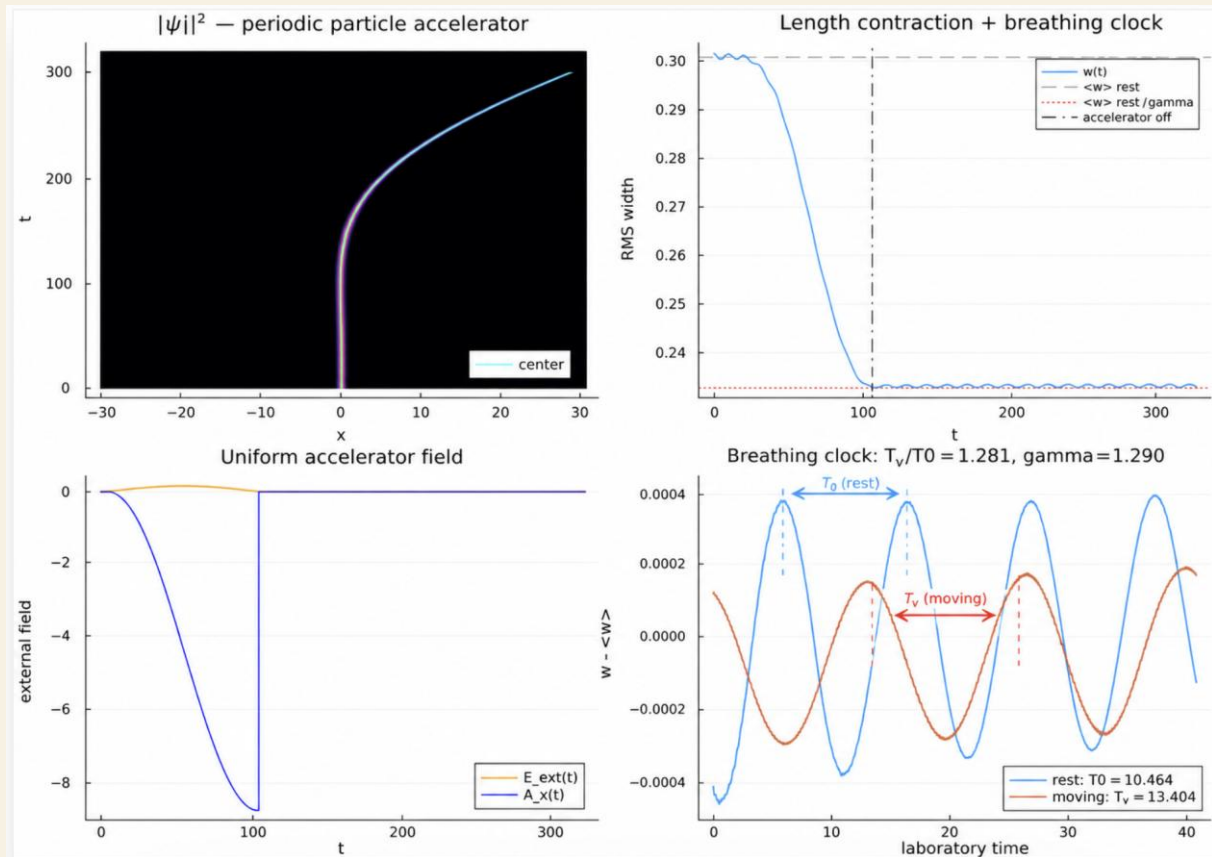
Here e is the coupling constant between the charged field and the electromagnetic field, and A_x is the electromagnetic vector potential from which the electric field is derived. In this setup, the electromagnetic field is imposed externally, so we do not consider its own dynamical equation.

This coupling is equivalent to saying that at each point the complex field ψ evolves in a complex internal space whose orientation is rotated, and $eA_x(t)$ represents the angular difference between neighbouring sites. In Part 3, we chose a different coupling to represent an electric field: the temporal coupling $\frac{\partial}{\partial t} - ie\Phi$. The complex internal space was represented by rotating platforms, and $e\Phi$

corresponded to their angular velocity. An electric field was present when neighbouring sites did not rotate at the same rate, that is, when $\frac{\partial \Phi}{\partial x} \neq 0$. Here we make a different choice: $A_x(t)$ represents only an angular offset of the complex "platform" between neighbouring sites. But an electric field still corresponds to a difference in the rate at which neighbouring platforms rotate. The condition for the presence of an electric field is therefore now $\frac{\partial A_x}{\partial t} \neq 0$. Two different choices of potential can thus lead to the same physical result: a difference in the rotation rate of the internal space of the field ψ between neighbouring points. This equivalence illustrates an idea that we will explore in detail in the next chapter and that lies at the heart of modern theoretical physics: local gauge invariance. For readers who remember their electromagnetism, the electric field is indeed given by $E_x(x, t) = -\frac{\partial \Phi(x, t)}{\partial x} - \frac{\partial A_x(x, t)}{\partial t}$, so an electric field can be produced through Φ , through A_x , or through both.

This electromagnetic field accelerates our soliton, and after some time we gradually switch the field off. It is important to turn the field on and off very smoothly so as not to excite the internal degrees of freedom of the soliton; otherwise it could deform for reasons unrelated to relativistic effects. This is called adiabatic acceleration. Once the field is off, the soliton continues moving at a constant relativistic speed. We then measure its width and breathing period and compare them with their rest values. That is the experimental protocol.

Now for the experiment. In this first run, after acceleration the soliton reaches a speed close to 60% of the speed of light. The image below shows the results of my Julia simulation:

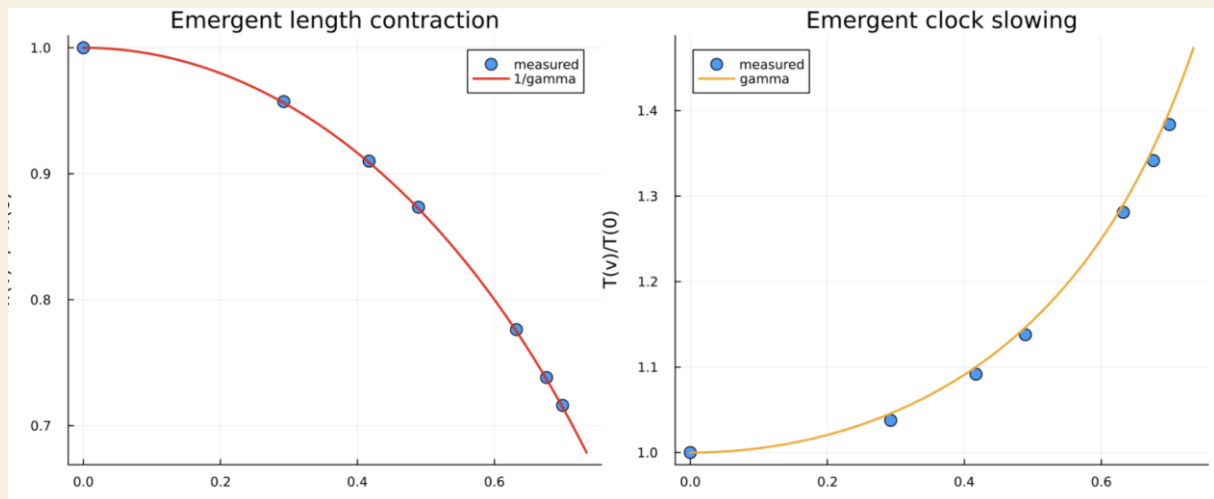


In the upper-left panel, we see the soliton accelerate and then continue along a straight trajectory. In the lower-left panel, the yellow curve represents the electric field.

In the upper-right panel, we see our first result: the width of the soliton decreases from 0.30 to about 0.23. The red dashed line is the theoretical prediction from Lorentz length contraction. The agreement between the simulation and the relativistic prediction is excellent.

The lower-right panel shows the breathing rhythm. The blue curve corresponds to the soliton at rest and the red curve to the moving soliton. The theoretical relativistic ratio between the two periods is 1.290; the simulation gives 1.281. Once again, the result is close to the expected value.

We now repeat the experiment several times for different values of the electromagnetic field. Here are the results:



On the left is the length-contraction factor as a function of speed. The blue points are simulation results, while the red curve is the theoretical value predicted by relativity under ideal conditions. Once again, the agreement is excellent. On the right are measurements of the clock-slowing factor as a function of speed, compared with the theoretical relativistic prediction under ideal conditions. Although these results are slightly less precise than the length-contraction results, they are still good. It is understandable that they are not perfect: the periods and lengths are estimated from finite simulation data, the acceleration is probably not perfectly adiabatic, and the spatial and temporal numerical resolution is finite. Nevertheless, the results remain convincing: they show that relativistic effects in structures such as solitons can be understood as consequences of the dynamics of the fields themselves.

An interactive animation also lets you visualize the length contraction and the slowing of the soliton's internal clock: [see the animation on the presentation page](#).

This should not surprise us: the equations we chose have the required relativistic symmetries; they are Lorentz invariant. It is therefore not mysterious in itself that they produce relativistic dynamical effects. The more remarkable point, however, is the following: as we saw in the previous part, these dynamical equations can be obtained as continuum limits of mechanical models that are not relativistic (springs, rotating platforms, and so on) and that, moreover, admit a preferred frame. Here, only the choice of interaction between the charged complex field and the real mediator field has not been made explicit in mechanical terms, although it is not impossible to do so. But the important point conceptually is not that these models can be mechanically realized. The important point is that the relativistic character of the phenomena appears as emergent. This way of viewing things demonstrates a possibility that is not ruled out by current physical data: the relativistic nature of our most advanced physical theories could itself be emergent, and experiments at very high energies

might provide evidence for this. In the same way, the Julia simulation we have just studied would fail to maintain agreement with relativity if we pushed the speed to a value extremely close to the speed of light.

Conversely, the absence so far of any disagreement with relativistic predictions in high-energy experiments tells us that Lorentz symmetry holds to extremely high precision up to very high energies. It may also be an exact symmetry of nature, at least in flat spacetime. We do not know. What this shows is that, at the fundamental level, the absence of a preferred frame is not presently forced upon us as a direct empirical fact about every conceivable microscopic description. At the same time, existing physics explains very well why any preferred frame, if one exists, should be extraordinarily difficult to detect.

The next chapter will be the final chapter devoted to classical field physics. It will focus on gauge theories. We will gradually build a genuine miniature "classical Standard Model": first incorporating the electromagnetic field, then the Higgs field. With the Higgs field, we will examine the mechanism responsible for generating the masses of fundamental particles: spontaneous symmetry breaking.

Still remaining within a classical framework, we will also examine something central to quantum field theory: the dependence of particle charges and masses on the energy scale at which they are probed, described by the renormalization group. This will provide another example of effective laws emerging from a more fundamental physics that, for now, remains unknown.